Honors Theory, Spring 2002, Yap
Homework 7

This is due on Mon April 29.

1. (10 Points) Let $w = (01)^n$ be a string. Prove that $K(\langle w \rangle) \le \ell(n) + C$ for some C, and $K(\langle w \rangle | n) = O(1)$.

   SOLUTION: Given input n, we can compute $(01)^n$. Thus $K(\langle (01)^n \rangle) \le \ell(n) + C$. This alsow shows that $K(\langle w \rangle | n)$ is $O(1)$.

2. (20 Points) Let $A \subseteq \mathbb{N}$ be any set. Let $\chi_A = b_0 b_1 b_2 \cdots$ be the ω-string such that $b_i = 1$ iff $i \in A$. Write $\chi_A[i:j]$ for $b_i b_{i+1} \cdots b_j$, and $\chi_A[j]$ for $\chi_A[0:j]$.
   (i) For all recursive A, there exists a constant $c = c(A)$ such that $K'(\chi_A[n]) \le c$. NOTE: $K'(x)$ here is $K(x|\ell(x))$, the length-conditioned Kolmogorov complexity function.
   (ii) For all r.e. A, there is a constant $c = c(A)$ such that $K'(\chi_A[n]) \le \ell(n) + c$.

   SOLUTION:

   (i) Since this is length-conditioned, we must show how to compute $\chi_A[n]$ when we are only given n. This is easy. Construct a STM M that on input n, runs some decisive Turing machine for A on the inputs $i = 0, 1, \ldots, n$ in order. Then M output $\chi_A[n]$.

   (ii) In case A is r.e., the previous Turing machine M is now modified to N as follows: N checks that its input has the form $\langle n, m \rangle$ with $m > n + 1$. If not, it rejects. Otherwise, it dovetails the computations of an acceptor for A on the inputs $i = 0, 1, \ldots, n$. When m of these computations accepts, then N outputs a string $b_0 b_1 \cdots b_n$ where $b_i = 1$ iff the acceptor for A has accepted i. If less than m of these computations accept, N will loop. Clearly, if $\chi_A[n]$ has m 1's, then our machine N on $\langle n, m \rangle$ will output $\chi_A[n]$. Thus, $K_N(\chi_A[n]|n) = \ell(m) \le \ell(n) + C$.

   REMARKS: The proof of (ii) is very revealing about the non-constructiveness of $K(x|y)$. To show $K(x) = \ell(z)$ where z is minimal program for x, we need not know how to construct z. We only have to show its existence! The m in the above proof is also purely existential.

3. (15 Points) Show that there is some C_0 such that every minimal programs is C_0-incompressible. More precisely, this says: for any $x \in \mathbb{N}$, if z is a minimal program for x (*i.e.*, $\Phi(z) = x$ and $K(x) = \ell(z)$) then $K(z) \ge \ell(z) - C$.
   HINT: Consider the function $f(z) = \Phi(\Phi(z))$. If there is no such C_0, then for every C, we can find x such that $K(x) - K_f(x) > C$.

   SOLUTION: Suppose no such C_0 exists. Using the HINT, we know that for each C, there is a $x = x(C)$ such that $K(x) - K_f(x) > C$. This contradicts the universality of K.

4. (25 Points) Recall the lower bound proof that the time-space product is $\Omega(n^2)$ for any deterministic Turing machine that accept palindromes. Prove the same result when Turing machine is nondeterministic.

   SOLUTION: Recall the proof in the lecture notes. Where does the proof break down if M is now nondeterministic? Only in the definition of the "compatibility" of (w, S), where we simulated M on input w "modulo S". Now, compatibility is defined to mean that "*some* computation path of M will produce the crossing sequence S". We may verify that Lemma 11 still holds with this new definition.