Honors Theory, Spring 2002, Yap
Homework 3
Due: Wed Feb 27 in class

MOSTLY ABOUT RECURSIVELY ENUMERABLE SETS In the following, we will identify the set $\{0,1\}^*$ with $\mathbb{N}$ via the dyadic notation. So we will interchangeably talk of (natural) numbers and binary strings.

1. (20 Points) In the notes on the Bernstein-Schröder theorem we showed a fixed point A^* for any monotone map $\mu : 2^X \to 2^X$. This fixed point was based on the idea of sets $A \subseteq X$ that are **small** in the sense that $A \subseteq \mu(A)$. (The set A is "small" relative to its image $\mu(A)$.) Now define a set $B \subseteq X$ to be **big** if $\mu(B) \subseteq B$. Show a fixed point B^* for μ based on big sets.

   Extra Credit: Construct an example in which the $B^* \neq A^*$.

   GENERAL COMMENT: Extra credit work is always optional, and should only be attempted after you solved the other problems. The points awarded depend on the nature of your solution.

2. (15 Points) Prove that a set $A \subseteq \mathbb{N}$ is r.e. if and only if there exists a recursive set $B \subseteq \{0,1,\#\}^*$ such that $A = \{w \in \{0,1\}^* : (\exists y \in \{0,1\}^*)[w\#y \in B]\}$.

3. (15+15+10 Points) Fix a deterministic universal Turing machine U such that $K(U) = RE|\{0,1\}$. We can view this U as computing a function $\Phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, where the output alphabet of U is assumed to be $\{0,1\}$ and we identify the set $\{0,1\}^*$ with $\mathbb{N}$ via the dyadic notation. Recall how transducers define functions – on input $\langle i, w \rangle$, if v is the non-blank word that is being scanned by the work head when U enters the accept state q_A, then $\Phi(i,w)$ is the dyadic number v. If $U(i,w)\uparrow$, then $\Phi(i,w)$ is undefined. Define
$$\Phi := \{\phi_i : i \in \mathbb{N}\}$$
   where $\phi_i : \mathbb{N} \to \mathbb{N}$ be the function $\phi_i(w) = \Phi(i,w)$. We define a function f to be **partial recursive** if $f \in \Phi$. Thus Φ is the function analogue of the class RE. Let sets $W_i := \{w \in \mathbb{N} : \phi_i(w)\downarrow\}$ and $E_i := \{\phi_i(w) : w \in \mathbb{N}, \phi_i(w)\downarrow\}$. Thus, W_i, E_i are basically the domain and range of ϕ (Mnemonic: ϕ_i is a map from the "west" set W_i to the "east" set E_i.) Prove the following:
   (i) A set $A \subseteq \mathbb{N}$ is r.e. iff $A = W_i$ for some i.
   (ii) A set $B \subseteq \mathbb{N}$ is r.e. iff $B = E_i$ for some i. HINT: to show that E_i is r.e. you need to "dovetail" together a denumerable sequence of computations.
   (iii) The set $TOT := \{i \in \mathbb{N} : \phi_i \text{ is total}\}$ is not r.e.