Final Exam (with Solutions)
Computational Geometry and Modeling
G22.3033.007, Spring 2005, Professor Yap

May 18, 2005

Instructions

- Out: May 4, 2005
- Due: May 6, noon.
- You may refer to any text books and web resources. But you may not consult with anyone. Unless the information is generic, you should give references.
- You are welcome to use any software (Core Library, Maple, etc) but please include source programs you use.

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1. Question 1. Let $A(X,Y) \in K[X,Y]$ where $K \subseteq \mathbb{C}$ is a field. Define $A = A(X,Y)$ to be **square-free** if A is non-zero and there does not exist $B \subseteq K[X,Y]$ such that $\deg(B) > 0$ and B^2 divides A. This definition seems to depend on the choice of K, not just on $A(X,Y)$. E.g., if $A(X,Y)$ is square-free when viewed as an element of $K[X,Y]$, we may be able to choose larger field K' containing K and show that $A(X,Y)$ is no longer square-free when considered as an element of $K'[X,Y]$. The following shows that this cannot happen. This result is important for us because the curve $A = 0$ to said to be **reduced** if A is square-free.
(a) Let A_x and A_y denote the derivative of A with respect to X and Y, respectively. Prove that if A is not square-free iff $\deg(\mathtt{GCD}(A, A_x)) > 0$, iff $\deg(\mathtt{GCD}(A, A_y)) > 0$.
(b) Conclude that the notion of square-freeness of A does not depend on the choice of K.

SOLUTION: This is worth 10+5 points.

(a) Suppose A is not square-free. Then $A = B^2C$ for some non-zero $B, C \in K[X,Y]$ and $\deg(B) > 0$. Then $A_x = B(2B_xC + BC_x)$ and so B divides both A_x and A. Hence B divides $\mathtt{GCD}(A, A_x)$. This proves $\deg(\mathtt{GCD}(A, A_x)) > 0$.

Conversely, suppose $\deg(\mathtt{GCD}(A, A_x)) > 0$. Then $A = GB$ for some nonzero B and $G = \mathtt{GCD}(A, A_x)$. Now, either $\deg_X(G) > 0$ or $\deg_Y(G) > 0$. Suppose $\deg_Y(G) > 0$. Then $A_y = G_yB + GB_y$ where $G_y \neq 0$. Since G divides A_y, we conclude that G must divide G_yB. But $\deg_Y(G) > \deg_Y(G_y)$ implies $\deg(\mathtt{GCD}(G, B)) > 0$. If $g = \mathtt{GCD}(G, B)$, we conclude that g divides G and B and so g^2 divides $A = GB$. This proves that A is not square-free.

(b) Note that three operations (i) $A \mapsto A_x$ (differentiation), (ii) $\mathtt{GCD}$ operation, and (iii) $B \mapsto \deg(B)$ (degree function) do not depend on the choice of the field K. Hence the predicate $\deg(\mathtt{GCD}(A, A_x)) > 0$ is also independent of the choice of K. But this predicate characterizes square-freeness, by part (a). ■

2. Question 2. When is a curve irreducible? Ruppert's criterion for (absolute) irreducibility of a bivariate polynomials is given in the appendix below. It is known that the curve $Y^n + X^{n+1} = 0$ is irreducible for all $n \geq 1$. Using Ruppert's criterion, show that this is the case when $n = 2$. HINT: If you work by hand, you should organize your work carefully, as the matrix can be as large as 24×13. EXTRA CREDIT if you can prove that this curve is irreducible for all n.

SOLUTION: This is worth 12 points.

To determine if $f(X, Y) = Y^2 + X^3$ is irreducible, we must find $g(X, Y), h(X, Y)$ such that $\deg(g) \leq (2, 2)$ and $\deg(h) \leq (3, 0)$, and the equation $f_Y g - f g_Y = f_X h - f h_X$ holds.

Let $g = \sum_{i=0}^{2} \sum_{j=0}^{2} g_{ij} X^i Y^j$ and $h = \sum_{i=0}^{3} h_i X^i$.

Let $\mathbf{X} = (1, X, Y, X^2, XY, Y^2, X^3 \cdots X^5 Y^3)$ (length 24) and $\mathbf{g} = (g_{00}, g_{10}, g_{01}, g_{20}, g_{11}, g_{02}, \ldots, g_{22}, h_0, \ldots, h_3)^T$ (length 13). Then we set up the 24×13 matrix M such that $\mathbf{X} M \mathbf{g} = 0$.

The matrix M is the following:

	g_{00}	g_{01}	g_{02}	g_{10}	g_{11}	g_{12}	g_{20}	g_{21}	g_{22}	h_0	h_1	h_3	h_3
X^5Y^3													
X^4Y^3													
X^3Y^3													
X^2Y^3													
XY^3													
X^5Y^2													
X^4Y^2													
X^3Y^2													
X^2Y^2								-1					-3
XY^2					-1							-2	
Y^2		-1									-1		
X^5Y									2				
X^4Y						2							
X^3Y			2										
X^2Y							-2						
XY				-2									
Y	-2												
X^5								1					
X^4					1							1	
X^3		1									2		
X^2										3			
X													
1													

The rows are labeled by monomials and columns are labeled by the coefficients of f and g. For clarity, we have omitted all zero entries. It is easy to check (visually) that M has full rank, i.e., rank 13. This is because all non-zero rows belong to one of the following two types: (1) the row has a nonzero entry that is is the only nonzero entry in its column. (2) the row can be paired with another row so that, together, both rows share non-zero positions with each other (and with no one else). Such pairs also have rank 2. Since there are 13 nonzero rows, the rank of M is 13. Hence the f is irreducible.

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3. Question 3. We have emphasized that we mainly want to study curves in real affine space. But consider

the following two circles C, C' of radii $r > 0$ and $R > 0$:

$$C : X^2 + Y^2 = r^2, \qquad C' : X^2 + Y^2 = R^2$$

viewed as curves in *projective complex space*, $\mathbb{P}^2(\mathbb{C})$. Determine as much as you can about the nature (topology, dimension, etc) of their intersection. HINT: first homogenize the above equations.

Determine as much as you can about the nature

SOLUTION: This is worth 8 points.

After homogenization with variable W, we get

$$C : X^2 + Y^2 = W^2 r^2, \qquad C' : X^2 + Y^2 = W^2 R^2.$$

If $W \neq 0$, clearly, C and C' do not intersect. Thus, they intersect in the set $X^2 + Y^2 = 0, W = 0$ (this set at infinity is called the "absolute"). For any complex x, there are two complex values $y = \pm \mathbf{i} x$ such that (x, y) lies on the absolute. Each point (x, y) where $x \neq 0$ corresponds to the point $(x : y : 0)$ in $\mathbb{P}^2(\mathbb{C})$. Thus the intersection of C and C' consists of two points, $(1 : \pm\mathbf{i} : 0)$.

Kinda Anna Saddi also did a nice computation showing the resultant of $X^2 X^2 + Y^2 - W^2 r^2$ and $X^2 + Y^2 - W^2 R^2$ with respect to Z is

$$(r^4 - 2r^2 R^2 + R^2)(X^2 + Y^2)^2 = (r^4 - 2r^2 R^2 + R^2)(X + \mathbf{i}Y)^2 (X - \mathbf{i}Y)^2$$

which clearly shows that the points $(1 : \pm\mathbf{i} : 0)$ are double points. Thus there are four intersection points, counted with multiplicity. This is what we expect from the strong Bezout's theorem.

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4. Question 4. In the Delaunay Tetrahedralization of a set $S \subseteq \mathbb{R}^3$ of n points, we usually assume that the set S is **non-degenerate** in the sense that no 4 points are co-planar and no 5 points are co-spherical. This is certainly true of Edelsbrunner's book. Suppose we now allow degenerate point sets.
(a) Define the **generalized Delaunay face** (gDf) of S to be the convex hull of a maximal set of 3 or co-planar points that lie on the boundary of an empty circle. (Empty means no points of S lie in the interior of this circle.) Similarly, a **generalized Delaunay tetrahedron** (gDt) of S to be the convex hull of a maximal set of 4 or more co-spherical points that lie on the boundary of an empty sphere. Describe in detail a data structure to represent this kind of generalized Delaunay tetrahedralization.

   HINT: To ensure that you can properly navigate such a structure, it is best to use a generalization of the half-edge data structure for planar graphs.

   (b) Generalize the flipping algorithm (Edelsbrunner, Chapter 5.4) to deal with this situation. Prove the correctness of your algorithm.

SOLUTION: This is worth 10+10 Points.

Part (a) We may represent the generalized Delaunay tetrahedralization (gDt) this by a kind of generalized half-edge structure. As a general goal we want to ensure (1) linear storage size and (2) constant time navigation among adjacent features. The half-edge data structures achieves these goals in 2-D.

The question we address is "What is the generalization of half-edges?" One answer is this: half-edges represent the incidence between a face and an edge. In 2-D, since each edge has 2 such incidences, we get 2 half-edges. In 3-D, we ought to get a circular list of such incidences for each edge. But more generally, we consider incidences between i-cells with $(i+1)$-cells. In 3-D, an i-cell is called a vertex, edge, face, region when $i = 0, 1, 2, 3$ (respectively). Thus, in 3-D, we also have face-region incidences. Each face has 2 such incidences, and so we have 2 half-faces. Thus, each edge e may be associated with a circular list of half-edges corresponding to incidences between e and a face f. Also, each half-edge (e, f) has a natural "next" half-edge of the form (e', f) for some e'. Write "gDf" ("gDr") for generalized Delaunay face (region).

Putting these together, we consider 4 linked lists:

- (1) Vertex List: one for each of the input points. We store the coordinates of each point with the vertex.
- (2) Edge List: each edge e has two fields:
  (2.1) An array $e.vertex[0..1]$ of two points. The direction from $e.vertex[0]$ to $e.vertex[1]$ the "canonical orientation" of e, and it is arbitrary.
  (2.2) A circular list $e.fanPtr$ of pointers to half-edges that "uses" edge e. We call this circular list the **fan** of e. To go around the fan, we either $e.fanPtr.next$ and $e.fanPtr.prev$ to go forward or backward around the fan. By repeated following of the *next* pointers, we move around the fan in a clockwise order as we look in the direction from $e.vertex[0]$ to $e.vertex[1]$.
- (3) Halfedge List: Each half edge h has four fields:
  (3.1) $h.edgePtr$ points to the edge associated with h.
  (3.2) $h.orient$ is 0 or 1, depending on whether the orientation of h is consistent or opposite of the canonical orientation of $h.edge$.
  (3.3) $h.hfacePtr$ points to the half face that uses this half-edge.
  (3.4) $h.fanLoc$ points to the position of h position in fan of $h.edgePtr$. Using this pointer, we can get the "next half face" around $h.edge$.
  (3.5) $h.ringLoc$ points to the position of h in the ring of $h.hfacePtr$.
- (4) Face List: one for each generalized Delaunay face (gDf). Each gDf f has these fields:
  (4.1) $f.normal$ a normal to the plane of f The normal direction is arbitrary up to sign.
  (4.2) $f.ringPtr$ points to the circular list of half-edges that uses face f. This structure is call the **ring** of f. Again, we can go forward or backward around *ringPtr* using $f.ringPtr.next$ or $f.ringPtr.prev$. This ring appears in the clockwise direction when the face is viewed along the $f.normal$ direction.
  (4.3) $f.gDrInner$ and $f.gDrOuter$ points to the two gDr's on each side of f; the inner gDr is on the side of f that $f.normal$ points to.
- (5) Region List: one for each generalized Delaunay region (gDr). Each gDr r has these fields:
  (5.1) $r.center$ the center of the circumsphere of this region;
  (5.2) $r.face$ points to one face bounding r.

For instance, using this data structure, we can traverse every edge incident on a given vertex in linear time. How?

Part (b). Let us figure out what flipping would look like in 2-D: if f is a gDf in 2-D, and p is a point in the circle of f, then we simply connect p to all the vertices of f and check if these triangles involving p are empty. We propagate from there. Suppose f has $k \geq 3$ vertices. There are actually two generic possibilities: if p lies inside the convex hull of the cocircular points of f, then we create k triangles and otherwise, we only create $k - 1$ triangles. But we could also have a degenerate posibility where p lies on an edge of the convex hull. But we can treat this as if p lies outside the convex hull.

The same thing happens in 3-D: if t is a gDr, and p is a point inside the sphere of the gDr, we connect p to each of the faces of the gDr to form generalized tetrahedras (cone tetrahedras). Again, if t has $k \geq 4$ faces, we have two possibilities: if p lies inside the convex hull the vertices, we create k tetrahedras, and otherwise only $k - 1$ tetrahedras. If p lies on a face of t, we treat it as the latter case.

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5. Question 5. Suppose α is an algebraic number, say, represented by the isolated interval representation. Let $B(X)$ and $C(X)$ be integer polynomials.

(i) Describe an algorithm to test if α is a zero of $B(X)$.
(ii) Describe an algorithm to test if $C(\alpha)$ is a zero of $B(X)$.

SOLUTION: This is worth 15+5 points.

(a) Let α has isolated interval representation $(A(X), I)$. Thus $A(\alpha) = 0$. Here are two ways to test if $B(\alpha) = 0$.

METHOD 1: We first compute the Sturm Sequence of B. Let $V_B(I)$ denote the number of sign variations of the Sturm sequence over I. If $V_B(I) = 0$ then clearly our test is negative (i.e., $B(\alpha) \neq 0$). What if $V_B(I) > 0$? We can only be sure that the test is positive (i.e., $B(\alpha) = 0$) if (1) $V_B(I) = 1$ and (2) $|I|$ is less than the root separation bound between zeros of A and zeros of B. Let $\Delta > 0$ be this separation bound (it is easily computed from A, B, as my lecture notes).

Thus we execute the following loop as long as $|I| \geq \Delta$: at each step, we bisect I and update I to be the left or right half of I which contains α. Compute $V_B(I)$. If $V_B(I) = 0$, we return negative, else continue the loop.

Thus the only way to exit the loop is when $|I| < \Delta$. At this point, we simply return positive (we can be sure that $V_B(I) = 1$). There is one small detail: what if the bisection "accidentally" chooses the bisection point that turns to be the root α? When in this case we can directly test if $B(\alpha) = 0$.

REMARK: for your answer, you ought to work the above remarks out into an actual algorithm in pseudo-code. This should be simple.

METHOD 2: Note that a necessary condition for $B(\alpha) = 0$ is that $\deg(\texttt{GCD}(A, B)) > 0$. What else do we need to ensure that the test "$B(\alpha) = 0$" is positive? The answer is to compute the Sturm Sequence for $G = \texttt{GCD}(A, B)$. Then we just need to check that the sign variation of $V_G(I) = 1$. Of course, if $V_G(I) = 0$, we know that the test is negative. What if $V_G(I) > 1$? Well, we enter a loop just as in METHOD 1 to narrow I down until we reach $V_G(I) \leq 1$.

(b) This is easily reduced to the previous case: simply replace B by the polynomial $B(C(X))$.

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Notes on Irreducibility Test

Recall that a polynomial $A(X,Y) \in K[X,Y]$ ($K \subseteq \mathbb{C}$) is said to be **absolutely irreducibile** if it has no factors in $C[X,Y]$. We simply say "irreducible" for "absolute irreducible". We have the following irreducibility criteria from Ruppert. First, write $\deg(A) \leq (m,n)$ if $\deg_X(A) \leq m$ and $\deg_Y(A) \leq n$.

THEOREM 1 *Let* $\deg(A) \leq (m,n)$. *A is irreducible iff*

$$\frac{\partial}{\partial Y}\frac{g}{f} - \frac{\partial}{\partial X}\frac{h}{f} = 0 \tag{1}$$

does not have a solution in $g, h \in \mathbb{C}[X,Y]$ *with* $\deg(g) \leq (m-1, n)$ *and* $\deg(h) \leq (m, n-2)$.

First note that (1) is equivalent to

$$f\frac{\partial g}{\partial Y} - g\frac{\partial f}{\partial Y} = f\frac{\partial h}{\partial X} - h\frac{\partial f}{\partial X}. \tag{2}$$

Note that the choice

$$g = \partial f/\partial X, \qquad h = \partial f/\partial Y$$

would satisfy (2) but this $\deg(h)$ does not satisfy the bounds on degrees of g and h in this theorem. Writing

$$F := f\frac{\partial g}{\partial Y} - g\frac{\partial f}{\partial Y} - f\frac{\partial h}{\partial X} + h\frac{\partial f}{\partial X}$$

we see that $\deg(F) \leq (2m-1, 2n-1)$. Let $\mathbf{x} = (X^{2m-1}Y^{2n-1}, X^{2m-2}Y^{2n-1}, \ldots, X, 1)$ denote a $(4mn)$-vector containing all power products that could appear F. Write $g = \sum_{i=0}^{m-1}\sum_{j=0}^{n} g_{ij}X^iY^j$ and $h = \sum_{i=0}^{m}\sum_{j=0}^{n-2} h_{ij}X^iY^j$ where g_{ij}, h_{ij} are indeterminates. There are $2mn + n - 1$ indeterminates, and we denote by $\mathbf{v}$ the $(2mn + n - 1)$-vector containing these indeterminates. Note that the F is linear in this set of determinates. Similarly, F is linear in the coefficients of g and h. Hence, we may write the equation $F = 0$ in the matrix form:

$$\mathbf{x}^T R(f)\mathbf{v} = 0$$

where $R(f)$ is a $(4mn) \times (2mn + n - 1)$ matrix whose entries are determined by the coefficients of f. Thus $F = 0$ is equivalent to $R(f)$ being singular. Then Ruppert's criterion says f is reducible iff $R(f)$ has rank less than $2mn + n - 1$.