Final Exam
Computational Geometry and Modeling
G22.3033.007, Spring 2005, Professor Yap

May 4, 2005

Instructions

- Out: May 4, 2005
- Due: May 6, noon.
- You may refer to any text books and web resources. But you may not consult with anyone. Unless the information is generic, you should give references.
- You are welcome to use any software (Core Library, Maple, etc) but please include source programs you use.

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1. Question 1. Let $A(X,Y) \in K[X,Y]$ where $K \subseteq \mathbb{C}$ is a field. Define $A = A(X,Y)$ to be **square-free** if A is non-zero and there does not exist $B \subseteq K[X,Y]$ such that $\deg(B) > 0$ and B^2 divides A. This definition seems to depend on the choice of K, not just on $A(X,Y)$. E.g., if $A(X,Y)$ is square-free when viewed as an element of $K[X,Y]$, we may be able to choose larger field K' containing K and show that $A(X,Y)$ is no longer square-free when considered as an element of $K'[X,Y]$. The following shows that this cannot happen. This result is important for us because the curve $A = 0$ to said to be **reduced** if A is square-free.
(a) Let A_x and A_y denote the derivative of A with respect to X and Y, respectively. Prove that if A is not square-free iff $\deg(\mathtt{GCD}(A, A_x)) > 0$, iff $\deg(\mathtt{GCD}(A, A_y)) > 0$.
(b) Conclude that the notion of square-freeness of A does not depend on the choice of K.

2. Question 2. When is a curve irreducible? Ruppert's criterion for (absolute) irreducibility of a bivariate polynomials is given in the appendix below. It is known that the curve $Y^n + X^{n+1} = 0$ is irreducible for all $n \geq 1$. Using Ruppert's criterion, show that this is the case when $n = 2$. HINT: If you work by hand, you should organize your work carefully, as the matrix can be as large as 24×13. EXTRA CREDIT if you can prove that this curve is irreducible for all n.

3. Question 3. We have emphasized that we mainly want to study curves in real affine space. But consider the following two circles C, C' of radii $r > 0$ and $R > 0$:
$$C : X^2 + Y^2 = r^2, \qquad C' : X^2 + Y^2 = R^2$$
viewed as curves in *projective complex space*, $\mathbb{P}^3(\mathbb{C})$. Determine as much as you can about the nature (topology, dimension, etc) of their intersection. HINT: first homogenize the above equations.

4. Question 4. In the Delaunay Tetrahedralization of a set $S \subseteq \mathbb{R}^3$ of n points, we usually assume that the set S is **non-degenerate** in the sense that no 4 points are co-planar and no 5 points are co-spherical. This is certainly true of Edelsbrunner's book. Suppose we now allow degenerate point sets.
(a) Define the **generalized Delaunay face** of S to be a maximal set of 3 or co-planar points that lie on the boundary of an empty circle. (Empty means no points of S lie in the interior of this circle.)

Similarly, a **generalized Delaunay tetrahedron** of S to be a maximal set of 4 or more co-spherical points that lie on the boundary of an empty sphere. Describe in detail a data structure to represent this kind of generalized Delaunay tetrahedralization.
(b) Generalize the flipping algorithm (Edelsbrunner, Chapter 5.4) to deal with this situation. Prove the correctness of your algorithm.

5. Question 5. Suppose α is an algebraic number, say, represented by the isolated interval representation. Let $B(X)$ and $C(X)$ be integer polynomials.
(i) Describe an algorithm to test if α is a zero of $B(X)$.
(ii) Describe an algorithm to test if $C(\alpha)$ is a zero of $B(X)$.

Notes on Irreducibility Test

Recall that a polynomial $A(X,Y) \in K[X,Y]$ ($K \subseteq \mathbb{C}$) is said to be **absolutely irreducibile** if it has no factors in $C[X,Y]$. We simply say "irreducible" for "absolute irreducible". We have the following irreducibility criteria from Ruppert. First, write $\deg(A) \leq (m,n)$ if $\deg_X(A) \leq m$ and $\deg_Y(A) \leq n$.

THEOREM 1 *Let* $\deg(A) \leq (m,n)$. *A is irreducible iff*

$$\frac{\partial}{\partial Y}\frac{g}{f} - \frac{\partial}{\partial X}\frac{h}{f} \tag{1}$$

does not have a solution in $g, h \in \mathbb{C}[X,Y]$ *with* $\deg(g) \leq (m-1, n)$ *and* $\deg(h) \leq (m, n-2)$.

First note that (1) is equivalent to

$$f\frac{\partial g}{\partial Y} - g\frac{\partial f}{\partial Y} = f\frac{\partial h}{\partial X} - h\frac{\partial f}{\partial X}. \tag{2}$$

Note that the choice

$$g = \partial f/\partial X, \qquad h = \partial f/\partial Y$$

would satisfy (2) but this $\deg(h)$ does not satisfy the bounds on degrees of g and h in this theorem. Writing

$$F := f\frac{\partial g}{\partial Y} - g\frac{\partial f}{\partial Y} - f\frac{\partial h}{\partial X} + h\frac{\partial f}{\partial X}$$

we see that $\deg(F) \leq (2m-1, 2n-1)$. Let $\mathbf{x} = (X^{2m-1}Y^{2n-1}, X^{2m-2}Y^{2n-1}, \ldots, X, 1)$ denote a $(4mn)$-vector containing all power products that could appear F. Write $g = \sum_{i=0}^{m-1}\sum_{j=0}^{n} g_{ij}X^iY^j$ and $h = \sum_{i=0}^{m}\sum_{j=0}^{n-2} h_{ij}X^iY^j$ where g_{ij}, h_{ij} are indeterminates. There are $2mn + n - 1$ indeterminates, and we denote by $\mathbf{v}$ the $(2mn + n - 1)$-vector containing these indeterminates. Note that the F is linear in this set of determinates. Similarly, F is linear in the coefficients of g and h. Hence, we may write the equation $F = 0$ in the matrix form:

$$\mathbf{x}^T R(f)\mathbf{v} = 0$$

where $R(f)$ is a $(4mn) \times (2mn + n - 1)$ matrix whose entries are determined by the coefficients of f. Thus $F = 0$ is equivalent to $R(f)$ being singular. Then Ruppert's criterion says f is reducible iff $R(f)$ has rank less than $2mn + n - 1$.