SOLUTION TO MIDTERM

Problem 1 (Asymptotics) [15 points]
ANSWER: The ordered functions, in increasing order of growth:
(d) $n^{1/\lg n}$ (This is equal to 2! Take the log and see.)
(b) $\lg\lg n$
(h) $n\lg^2 n$
(f) n^2
(e) $2^{3\lg n}$ (This is equal to n^3)
(c) $n^{\lg\lg n}$ (To compare (c) and (g), take their logs)
(g) $(\lg\lg n)^n$
(a) $n!$ (By Stirling's formula, this is $\Theta((n/e)^{n+(1/2)})$

Problem 2 (Summations) [4+4+8 points]
(a) ANSWER: Recall (analysis of BuildHeap!) that $\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$. Plugging $x = 1/3$ into the formula, we get

$$\sum_{i=1}^{\infty} i(1/3)^i = \frac{(1/3)}{(1-(1/3))^2} = \frac{3}{4}.$$

(b) ANSWER: Using the hint, we see that $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$. Hence the sum "telescopes" into

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}.$$

(c) ANSWER: The two sums are

$$\sum_{i=0}^{2n} 3^i = \frac{3^{2n+1}-1}{3-1} = \Theta(9^n)$$

$$\sum_{i=0}^{n} 8^i = \frac{8^{n+1}-1}{8-1} = \Theta(8^n)$$

Clearly the former is asymptotically dominant.

Problem 3 (Recurrences) [15+8 points]
(a) ANSWER: Using the Master theorem, we get

$$\begin{aligned}
T_1(n) &= 100T_1(\frac{n}{100}) + n + \log^2 n \\
&= \Theta(n\log n) \\
T_2(n) &= 100T_2(\frac{n}{10}) + n + \sqrt{n} \\
&= \Theta(n^2) \\
T_3(n) &= 10T_3(\frac{n}{100}) + n + 3 \\
&= \Theta(n) \\
T_4(n) &= 100T_4(\frac{n}{100}) + n + n^2 \\
&= \Theta(n^2) \\
T_5(n) &= 100T_5(\frac{n}{10}) + n\log n + n^2 \\
&= \Theta(n^2\log n)
\end{aligned}$$

(b) Solve up to Θ-order (choose your own boundary conditions):

$$T(n) = 3T(n-2) + n$$

ANSWER: Let n be an even number. By a domain transformation, define $k = n/2$ and $S(k) = T(2k) = T(n)$. Then

$$S(k) = 3S(k-1) + 2k.$$

(You could also just use "repeated expansion".) By the range transformation, let $R(k) = \frac{S(k)}{3^k}$. Then, dividing the above recurrence by 3^k gives

$$R(k) = R(k-1) + \frac{2k}{3^k}.$$

This is in standard form! Telescoping, we get

$$R(k) = \sum_{i=1}^{k} \frac{2i}{3^i} \le 2\sum_{i=1}^{\infty} \frac{i}{3^i} = \frac{3}{2}$$

(where the last equation comes from problem 2(a)). [You could say that this sum is $\Theta(1)$, if you do not have time to figure out the exact number.] Thus $R(k) = \Theta(1)$ and $S(k) = 3^k R(k) = \Theta(3^k)$ and $T(n) = S(n/2) \le \Theta(3^{(n/2)})$.

Extra Credit Work: solve for $T(n)$ exactly, by working out the exact solution to the sum for $R(k)$ above. [You should speak to me before doing extra credit work.]

Problem 4 (Probability) [4+10 points]

(a) Ways to choose 7 dresses out of 30:

ANSWER: There 307 ways to choose 7 dresses. For each set of 7 dresses, you get to choose two dresses for the weekend, in 72 ways. The answer is therefore

30772.

(b) What is the expected profit if you pay \$7 to roll a pair of dice and your payoff is $\$(i+j)$ when the outcomes of the dice are i and j?

ANSWER: Following the hint, we fill out the table. The last row is the "weighted profit" (= profit times # of ways).

sum=i+j	:	2	3	4	5	6	7	8	9	10	11	12
profit	:	-5	-4	-3	-2	-1	0	1	2	3	4	5
# of ways	:	1	2	3	4	5	6	5	4	3	2	1
weighted	:	-5	-8	-9	-8	-5	0	5	8	9	8	5

The sum of the weighted profits (summing up the last row) is 0. In case you did not notice this, I should add that the third row (# of ways) is slightly tricky and you need to understand it. For instance, why is the # of ways to get a sum of 8 equal to 5? Because 8 can be obtain with (i, j) equal to any of

$$(2,6), \quad (6,2), \quad (3,5), \quad (5,3), \quad (4,4).$$

Problem 5 (Sorting) [4+4+4+4 points]

(a) Apply BUILD-HEAP to the following array $A = [5, 3, 17, 10, 11, 9]$.

(b) Now carry out the rest of the heapsort algorithm on the heap you just obtained.

(c) Do bubble sort on the original array (in part (a)).

(d) Do Insertion sort on the original array.

ANSWER:

(a)

```
      5             5             5              17
     / \           / \           / \            /  \
    3   17        3   17       11  17         11    9
   / \  /        / \  /        / \ /         / \    /
  10 11 9       10 11 9       10  3 9       10  3  5

  Initial     Heapify(17)  Heapify(3)   Heapify(5)
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(b) The array after each DeleteMax is shown (a vertical bar indicates the extent of the current heap):

$$17, 11, 9, 10, 3, 5|$$
$$11, 10, 9, 5, 3|17$$
$$10, 5, 9, 3|11, 17$$
$$9, 5, 3|10, 11, 17$$
$$5, 3|9, 10, 11, 17$$
$$3|5, 9, 10, 11, 17$$

(c) Bubble sort:

$$5, 3, 17, 10, 11, 9|$$
$$3, 5, 10, 11, 9|17$$
$$3, 5, 10, 9|11, 17$$
$$3, 5, 9|10, 11, 17$$
$$3, 5|9, 10, 11, 17$$
$$3|5, 9, 10, 11, 17$$

(d) Insertion sort:

$$5, 3, 17, 10, 11|9$$
$$5, 3, 17, 10|9, 11$$
$$5, 3, 17|9, 10, 11$$
$$5, 3|9, 10, 11, 17$$
$$5|3, 9, 10, 11, 17$$
$$3, 5, 9, 10, 11, 17$$

Problem 6 (Trees) [10 points]

Let n_i be the number of degree i nodes in a rooted tree. Prove by induction that

$$n_0 = 1 + \sum_{i \geq 2} (i-1) n_i.$$

PROOF: We use induction on the number n of nodes in the tree. (BASIS) This is true when $n = 1$. (INDUCTION) If we delete a leaf, we get a tree with 1 fewer node whose corresponding numbers n'_i satisfy the equation

$$n'_0 = \sum_{i \geq 2} (i-1) n'_i$$

(by induction hypothesis). Let x be the node whose child was deleted. We consider three cases, depending on the degree of x in the original tree. If x has degree 1 then we see that

$n'_i = n_i$ for $i = 0$ and for all $i \geq 2$. It is easy to see that the equation for n_i's is satisfied in this case. If x has degree $d \geq 2$, then

$$n'_0 = n_0 - 1, \quad n'_d = n_d - 1, \quad n'_{d-1} = n_{d-1} + 1$$

and all other n'_i is the same as n_i. First assume $d \geq 3$. Then

$$\begin{aligned}
n_0 &= n'_0 + 1 \\
&= 2 + \sum_{i \geq 2} (i-1) n'_i \\
&= 2 + (d-1)n'_d + (d-2)n'_{d-1} + \sum_{i \neq d, d-1} (i-1) n'_i \\
&= 2 + (d-1)(n_d - 1) + (d-2)(n_{d-1} + 1) + \sum_{i \neq d, d-1} (i-1) n_i \\
&= 1 + (d-1)n_d + (d-2)n_{d-1} + \sum_{i \neq d, d-1} (i-1) n_i \\
&= 1 + \sum_{i \geq 2} (i-1) n_i .
\end{aligned}$$

Please do the same calculation for $d = 2$ yourself. [Alternative proof: show that there is a node of positive degree all of whose children are leaves. Then delete all the children of this node.]