Greedy Drawings of Planar Triangulations

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We have a network of wireless sensors that need to send data to one another.

The network is *ad-hoc*.

**Simple Idea:** Use **Geographic Routing**.

- Each sensor knows its own location and the location of its neighbors.
- Source knows address of destination and encodes it in the packet.
- Each packet is sent to the neighbor closest to the destination. This is known as **Greedy Routing**.

But what if precise location of sensors is not known?
Routing in *Ad-Hoc* Networks

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- The network is *ad-hoc*.
- Simple Idea: Use **Geographic Routing**.
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  - Source knows address of destination and encodes it in the packet.
  - Each packet is sent to the neighbor closest to the destination. This is known as **Greedy Routing**.
- But what if precise location of sensors is not known?

**Solution:** [Rao et. al]

Use **Virtual Coordinates**.
Think of the wireless network as a graph, $G$.

Draw $G$ in the plane and pretend that the coordinates of the nodes in the drawing is the actual geographic location.

What property of the drawing is needed to ensure that greedy routing works?

**Greedy Routing works iff:**

For every pair of nodes $u$ and $v$, there exists a neighbor $u'$ of $u$ such that in the drawing, $\text{distance}(u, v) > \text{distance}(u', v)$.

Such a drawing is called a **Greedy Drawing**.

Is such a drawing always possible?
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- Such a drawing is called a **Greedy Drawing**.
- Is such a drawing always possible?

**NO!**
$K_{1,6}$ has no greedy embedding. More generally, $K_{n,5n+1}$ has no greedy embedding [PR05].
Examples of Graphs with Greedy Drawings

- Any graph with a Hamiltonian path.
- Any Complete Graph.
- Any four connected planar graph.
- Any Delaunay Triangulation.

What about:

- Three connected planar graphs?
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Conjectured by Papadimitriou-Ratajczak

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Our Result

Any planar triangulation has a greedy drawing.
Partitioning the edges of a triangulation: Schnyder’s Realizers.
What are Schnyder drawings of a triangulation?
Nice geometric properties of Schnyder drawings.
Connection between Schnyder drawings and Greedy drawings.
The Knaster-Kuratowski-Mazurkiewicz theorem.
Putting all this together, we can show that triangulations have greedy drawings.
Given a triangulation $G$, 

- Fix some face $f_0$ and call it the **external** face.
- All edges not belonging to this face are called the **internal** edges.
- Schnyder developed a method for partitioning the internal edges of $G$ into three directed trees rooted at the vertices of $f_0$.
- He also developed a way of obtaining a planar drawing of $G$ using this partitioning.
- A rather elegant result!
Schnyder’s Realizers: An Example

The Triangulation.
Schnyder’s Realizers: An Example

The Red Tree.
The Green Tree.
Schnyder’s Realizers: An Example

The Blue Tree.
Schnyder’s Realizers: An Example

All Together.
Properties of Schnyder Realizers

Order of edges at a vertex.
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Properties of Schnyder Realizers

Order of edges at a vertex.

3 directed paths exist from each vertex.
Obtaining a Schnyder Drawing
OBTAINING A SCHNYDER DRAWING

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Assign a non-negative weight $w_i$, to each internal face such that 
$\sum_i w_i = 1$
Let sum of weights in each of the three regions be $w_{\text{red}}$, $w_{\text{green}}$ and $w_{\text{blue}}$. 
The position of the vertex is given by \((w_{blue}, w_{green}, w_{red})\). This is a point on the \(x + y + z = 1\) plane.
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We can find such positions for every internal vertex.
The position of the vertex is given by \((w_{\text{blue}}, w_{\text{green}}, w_{\text{red}})\). This is a point on the \(x + y + z = 1\) plane.

What about external vertices?
Obtaining a Schnyder Drawing

The position of the vertex is given by \((w_{\text{blue}}, w_{\text{green}}, w_{\text{red}})\). This is a point on the \(x + y + z = 1\) plane.

[SCHNYDER 91]

We obtain a planar drawing of \(G\) on the \(x + y + z = 1\) plane.
How many Schnyder Drawings exist?

- Any set of non-negative face weights that sum to 1 can be used to obtain a Schnyder Drawing.
- Total number of faces $= 2n - 5$.
- Any solution to the equation:

$$w_1 + w_2 + \ldots + w_{2n-5} = 1, w_i \geq 0$$

gives a Schnyder Drawing.
- The set of all solutions to the above equation is the $2n - 5$ dimensional unit simplex $S_{2n-5}$.
- For each point $p \in S_{2n-5}$, we obtain a Schnyder Drawing of $G$. 

Every face has equal weight.
The weight of the Magenta-colored face is being increased.
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**Three Wedges Property**

Edge slopes always fall in a pre-determined $60^\circ$-wedge.
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**Schnyder Drawings: Geometric Properties**

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**Enclosing Triangle Property**

Outer triangle is the external face.
**Schnyder Drawings: Geometric Properties**

**Three Wedges Property**

Edge slopes always fall in a pre-determined $60^\circ$-wedge.

**Enclosing Triangle Property**

The Yellow Triangle is free of other nodes.
Solving “Half” the Problem

A half solution

Given any two vertices $u$ and $v$, in every Schnyder Drawing of $G$, either $u$ has a neighbor closer to $v$ or $v$ has a neighbor closer to $u$. 
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SOLVING “Half” THE PROBLEM

A half solution

Given any two vertices \( u \) and \( v \), in every Schnyder Drawing of \( G \), either \( u \) has a neighbor closer to \( v \) or \( v \) has a neighbor closer to \( u \).
Finding Greedy Paths
Finding Greedy Paths

![Diagram showing a triangle with vertices labeled u and v. Arrows indicate a greedy path from one vertex to another.]
Finding Greedy Paths

Greedy Drawings
Finding Greedy Paths

Jumping from $u$ to one of $u_1$ or $u_2$ gets you closer to $v$ in every Schnyder Drawing.

More Complicated!
Good & Bad Faces

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Good & Bad Faces

Type I

Type II
Let $\epsilon > 0$ be some constant. The face $(u, u_1, u_2)$ is said to be **good** if:

- Length of every edge of the face is at least $\epsilon$.
- Jumping from $u$ to $u_1$ gets you $\epsilon$-closer to all vertices of Type I.
- Jumping from $u$ to $u_2$ gets you $\epsilon$-closer to all vertices of Type II.

and **bad** otherwise.
Recall: For each point $p \in S_{2n-5}$, there exists a Schnyder Drawing of $G$.

For each internal face $f_i$, we define Good Set, $G_{f_i} \subseteq S_{2n-5}$ in the following way:

- Let $p \in S_{2n-5}$. Then $p \in G_{f_i}$ iff the face $f_i$ is good in the Schnyder Drawing corresponding to $p$. 
Recall: For each point \( p \in S_{2n-5} \), there exists a Schnyder Drawing of \( G \).

For each internal face \( f_i \), we define **Good Set**, \( G_{f_i} \subseteq S_{2n-5} \) in the following way:

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**When does a greedy drawing exist?**

Triangulation \( G \) has a greedy drawing iff \( \bigcap_{1 \leq i \leq 2n-5} G_{f_i} \neq \emptyset \).
Suppose we can show the following two properties:

- The good sets $\mathcal{G}_{f_i}$ are closed.
- In the drawing corresponding to any $p \in S_{2n-5}$, the sum of the weights of all the bad faces is strictly less than 1.

What can these two properties buy us?
Suppose we can show the following two properties:

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What can these two properties buy us?

**Surprise!**
These two properties are sufficient to show that a greedy drawing exists!
The KKM Theorem

**Theorem**

Given a simplex $S$ in $d$ dimensions, with vertices $v_0, v_1, \ldots, v_{d+1}$ and closed sets $G_1, G_2, \ldots, G_{d+1}$ such that the following condition is satisfied:

Any face of the simplex spanned by vertices $v_{j_1}, v_{j_2}, \ldots, v_{j_k}$ is covered by the sets $G_{j_1}, G_{j_2}, \ldots, G_{j_k}$.

Then: $\bigcap_{1 \leq i \leq d+1} G_i \neq \emptyset$.

- This can be used to show that greedy drawings exist.
- Crucial to show that the sum of weights of all bad faces is always strictly less than 1.
- This can be done using *Canonical Ordering* [dFPP90].
- We try to find a set of face weights such that the sum of weights of bad faces is 1 and arrive at a contradiction.
If shaded region is free of vertices, then only way the face can be bad is if the sides of the face are too short.
The only way this face can be bad is if the sides of the face are too short. If $\epsilon$ is small enough, the weight of this face must be small if it is to be bad.
Canonical Ordering of Vertices

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CANONICAL ORDERING OF VERTICES

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The End.
Any Questions?