Let $\mathcal{X}$ be a set and $\mathcal{L} \subseteq \wp(\mathcal{X})$ be a set of subsets of $\mathcal{X}$ (i.e. $\mathcal{L} \in \wp(\wp(\mathcal{X}))$) where $\wp(\mathcal{X}) \triangleq \{ Y \mid Y \subseteq \mathcal{X} \}$ and $(X \subseteq Y) \triangleq (\forall x \in X : x \in Y)$.

**Question 1** Consider the English sentence “$\mathcal{L}$ equipped with the partial order $\subseteq$ is a complete lattice (i.e. arbitrary least upper bounds do exist)”.

Formalize that English sentence in first order logic with equality (using symbols such as $\forall$, $\exists$, $\subseteq$, $\Rightarrow$, $\cup$, $\subseteq$, etc.).

**Question 2** According to your answer to Question 1, is the empty set $\emptyset$ a complete lattice?

**Question 3** Let $C$ be a set and $S \subseteq C$ be a non-empty subset of $C$ (i.e. $S \neq \emptyset$). Prove that $X \cap S \subseteq Y$ if and only if $X \subseteq Y \cup \neg S$ where $\neg S \triangleq C \setminus S$.

**Question 4** Let $C$ and $A$ be sets and $S \subseteq C$ be a non-empty subset of $C$ (i.e. $S \neq \emptyset$). Prove that $\alpha(X) \triangleq X \cap S$ is the lower adjoint of a Galois connection $(\wp(C), \subseteq) \xrightarrow{\gamma} (\wp(A), \subseteq)$.

In the following questions, we let $\langle L, \subseteq, \bot, \top, \cup, \cap \rangle$ be a complete lattice and $f \in L \mapsto L$ be an increasing function of $L$ into $L$. A bounded widening on $L$ is $\forall \in L \mapsto (L \times L) \mapsto L$ such that for all $S \in L$ (writing $x \searrow_S y$ for $\forall(S)(x, y)$ when $x, y, S \in L$):

- $\forall x, y \in L : (x \subseteq y \subseteq S) \Rightarrow (y \subseteq x \searrow_S y \subseteq S)$, and
- For any sequence $\langle y_n, n \in \mathbb{N} \rangle$, the sequence $x_0 \triangleq \bot, \ldots, x_{n+1} \triangleq x_n \searrow_S y_n, \ldots$ is ultimately stationary (that is $\exists \ell : \forall n \geq \ell : x_n = x_\ell$)

Define the iteration for $f$ and $S$ with bounded widening $\forall_S$ to be the sequence $\langle f^n, n \in \mathbb{N} \rangle$ such that $f^0 = \bot$ and $\forall n \in \mathbb{N} : f^{n+1} = f^n \searrow_S f(f^n)$ when $f^n \subseteq f(f^n) \subseteq S$ while $f^{n+1} = f^n$ otherwise.

**Question 5** Prove that the iteration $\langle f^n, n \in \mathbb{N} \rangle$ for $f$ with bounded widening $\forall_S$ is bounded by $S$ and increasing (i.e. $\forall n \in \mathbb{N} : f^n \subseteq f^{n+1} \subseteq S$).

**Question 6** Prove that the iteration for $f$ with bounded widening $\forall_S$ is ultimately stationary.

**Question 7** Let the iteration for $f$ with bounded widening $\forall_S$ be ultimately stationary at rank $\ell \in \mathbb{N}$. Prove that if $f(f^\ell) \subseteq f^\ell$ then $\text{lfp} \subseteq f \subseteq S$. 

\[1/1\]