Let $\mathcal{X}$ be a set and $\mathcal{L} \subseteq \wp(\mathcal{X})$ be a set of subsets of $\mathcal{X}$ (i.e. $\mathcal{L} \in \wp(\wp(\mathcal{X}))$ where $\wp(\mathcal{X}) \triangleq \{ Y \mid Y \subseteq \mathcal{X} \}$ and $(X \subseteq Y) \triangleq (\forall x \in X : x \in Y))$.

**Question 1** Consider the English sentence “$\mathcal{L}$ equipped with the partial order $\subseteq$ is a complete lattice (i.e. arbitrary least upper bounds do exist)”. Formalize that English sentence in first order logic with equality (using symbols such as $\forall$, $\exists$, $\subseteq$, $\Rightarrow$, $\cup$, $=$, etc.).

**Answer 1**

$$(\forall X \in \mathcal{L} : X \subseteq X) \land (\forall X, Y \in \mathcal{L} : (X \subseteq Y \land Y \subseteq X) \Rightarrow (X = Y)) \land (\forall X, Y, Z \in \mathcal{L} : (X \subseteq Y \land Y \subseteq Z) \Rightarrow (X \subseteq Z)) \land (\forall S \subseteq \mathcal{L} : \bigcup S \in \mathcal{L}).$$

The existence of a join/least upper bound on arbitrary subsets implies the existence of a meet/greatest lower bound.

**Question 2** According to your answer to Question 1, is the empty set $\emptyset$ a complete lattice?

**Answer 2** The empty set $\emptyset$ is not a complete lattice which must contains $\bigcup \emptyset$ hence is not empty.

**Question 3** Let $C$ be a set and $S \subseteq C$ be a non-empty subset of $C$ (i.e. $S \neq \emptyset$). Prove that $X \cap S \subseteq Y$ if and only if $X \subseteq Y \cup \neg S$ where $\neg S \triangleq C \setminus S$.

**Answer 3**

$$X \cap S \subseteq Y \iff (\forall x \in X : x \in Y) \iff (\forall x \in X : x \in Y \cup \neg S)$$

$$(\Rightarrow)$$ if $x \in X$ then either $x \in S$ so $x \in X \cap S$ implies $x \in Y$ hence $x \in Y \cup \neg S$ or $x \notin S$ so $x \in \neg S$ hence $x \in Y \cup \neg S$.

$$(\Leftarrow)$$ If $x \in X \cap S$ then $x \in X$ so $x \in Y \cup \neg S$ but $x \in S$ so $x \notin \neg S$ so $x \in Y$. 

$$X \subseteq Y \cup \neg S \iff (\forall x \in X : x \in Y \cup \neg S) \iff (\forall x \in X : x \in Y)$$
Question 4 Let $C$ and $A$ be sets and $S \subseteq C$ be a non-empty subset of $C$ (i.e. $S \neq \emptyset$). Prove that $\alpha(X) \triangleq X \cap S$ is the lower adjoint of a Galois connection $\langle \varphi(C), \subseteq \rangle \iff \frac{\gamma}{\alpha} \langle \varphi(A), \subseteq \rangle$. \hfill \Box

Answer 4 For all $X \in \varphi(C)$ and $Y \in \varphi(A)$, we have, according to the definition of a Galois connection,

\[
\alpha(X) \subseteq Y \\
\iff X \cap S \subseteq Y \quad \text{\{def. } \alpha \text{\}} \\
\iff X \subseteq Y \cup \neg S \quad \text{\{by Question 3\}} \\
\iff X \subseteq \gamma(Y) \quad \text{\{by defining } \gamma(Y) \triangleq Y \cup \neg S \text{\}} \hfill \Box
\]

In the following questions, we let $\langle L, \sqsubseteq, \bot, \top, \sqcup, \sqcap \rangle$ be a complete lattice and $f \in L \mapsto L$ be an increasing function of $L$ into $L$. A bounded widening on $L$ is $\nabla \in L \mapsto (L \times L) \mapsto L$ such that for all $S \in L$ (writing $x \nabla_S y$ for $\nabla(S)(x, y)$ when $x, y, S \in L$):

- $\forall x, y \in L : (x \sqsubseteq y \subseteq S) \Rightarrow (y \sqsubseteq x \nabla_S y \subseteq S)$, and (1)
- For any sequence $\langle y_n, n \in \mathbb{N} \rangle$, the sequence $x_0 \triangleq \bot, \ldots$, (2)

Define the iteration for $f$ and $S$ with bounded widening $\nabla_S$ to be the sequence $\langle f^n, n \in \mathbb{N} \rangle$ such that $f^0 = \bot$ and $\forall n \in \mathbb{N} : f^{n+1} = f^n \nabla_S f^n$ when $f^n \sqsubseteq f^n \nabla_S f^n \subseteq S$ while $f^{n+1} = f^n$ otherwise.

Question 5 Prove that the iteration $\langle f^n, n \in \mathbb{N} \rangle$ for $f$ with bounded widening $\nabla_S$ is bounded by $S$ and increasing (i.e. $\forall n \in \mathbb{N} : f^n \sqsubseteq f^{n+1} \subseteq S$). \hfill \Box

Answer 5 By recurrence. For the basis $f^0 = \bot$ either $f(\bot) \sqsubseteq S$ so, by definition of the infimum, $f^0 = \bot \sqsubseteq f(\bot) \sqsubseteq S$ hence $f^1 = f^0 \nabla_S f(f^0) \sqsubseteq S$ by (1) proving that $f^0 = \bot \sqsubseteq f^1 \subseteq S$. Otherwise, $f^1 = f^0 = \bot \subseteq S$ by definition of the iterates and the infimum so again $f^0 \sqsubseteq f^1 \subseteq S$.

For the induction step, assume that $f^n \sqsubseteq f^{n+1} \subseteq S$ by induction hypothesis. If $f^{n+1} \sqsubseteq f(f^{n+1}) \subseteq S$ then by (1), $f(f^{n+1}) \sqsubseteq f^{n+1} \nabla_S f(f^{n+1}) \subseteq S$ so that $f^{n+1} \sqsubseteq f^{n+2} \subseteq S$ by transitivity and definition of the iterates.

Otherwise $f^{n+2} = f^{n+1}$ so $f^{n+1} \sqsubseteq f^{n+2} \subseteq S$ by reflexivity and induction hypothesis. \hfill \Box

Question 6 Prove that the iteration for $f$ with bounded widening $\nabla_S$ is ultimately stationary. \hfill \Box

Answer 6 Either $\exists \ell \in \mathbb{N} : \neg(f^\ell \sqsubseteq f(f^\ell) \sqsubseteq S)$ and then, by definition of the iterates and recurrence, $\forall n \geq \ell : f^n = f^\ell$ so that $\langle f^n, n \in \mathbb{N} \rangle$ is ultimately stationary.

Else $\forall \ell \in \mathbb{N} : f^\ell \sqsubseteq f(f^\ell) \subseteq S$, and then by choosing $\forall n \in \mathbb{N} : x_n = f^n$ and $y_n \triangleq f(f^n)$, the condition (2) in the definition of the bounded widening implies that $\langle f^n, n \in \mathbb{N} \rangle$ is ultimately stationary. \hfill \Box

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Question 7 Let the iteration for $f$ with bounded widening $\forall S$ be ultimately stationary at rank $\ell \in \mathbb{N}$. Prove that if $f(f^\ell) \sqsubseteq f^\ell$ then $\text{lfp}^= f \sqsubseteq S$.

Answer 7 By Tarski’s theorem, $\text{lfp}^= f = \bigcap \{x \mid f(x) \sqsubseteq x\}$ and so $\text{lfp}^= f \sqsubseteq f^\ell$ since $f^\ell \in \{x \mid f(x) \sqsubseteq x\}$ and definition of a greatest lower bound. By Question 5, $f^\ell \sqsubseteq S$. By transitivity, we conclude that $\text{lfp}^= f \sqsubseteq S$.