Question 1 Consider the English sentence "L equipped with the partial order \( \subseteq \) is a lattice". Formalize that English sentence in first order logic with equality (using symbols such as \( \forall \), \( \exists \), \( \subseteq \), \( \Rightarrow \), \( \cup \), etc.).

Answer 1 \( (\forall X \in L : X \subseteq X) \land (\forall X, Y \in L : (X \subseteq Y \land Y \subseteq X) \Rightarrow (X = Y)) \land (\forall X, Y, Z \in L : (X \subseteq Y \land Y \subseteq Z) \Rightarrow (X \subseteq Z)) \land (\forall X, Y \in L : X \cup Y \in L) \land (\forall X, Y \in L : X \cap Y \in L) \).

Question 2 According to your answer to Question 1, is the empty set \( \emptyset \) a lattice?

Answer 2 Not clear from the English sentence, but \( \emptyset \) is a lattice since it satisfies the above definition, which is a conjunction of predicates of the form

\[
\forall X, Y, \ldots \in \emptyset : P(X, Y, \ldots)
\]

\[

\Leftrightarrow \forall X, Y, \ldots : (X \in \emptyset \land Y \in \emptyset \land \ldots) \Rightarrow P(X, Y, \ldots)
\]

\[
\Leftrightarrow \forall X, Y, \ldots : \text{false} \Rightarrow P(X, Y, \ldots) \quad \text{(since for all } X, X \in \emptyset \text{ is false)}
\]

\[
\Leftrightarrow \forall X, Y, \ldots : \text{true}
\]

\[
\Leftrightarrow \text{true}
\]

Question 3 Let \( C \) and \( A \) be sets. Let \( f \in C \mapsto \wp(A) \). Prove that \( \alpha(X) \triangleq \bigcup \{ f(x) \mid x \in X \} \) is the lower adjoint of a Galois connection \( \langle \wp(C), \subseteq \rangle \xrightarrow{\gamma} \langle \wp(A), \subseteq \rangle \).
\[ \alpha(X) \subseteq Y \]
\[ \Leftrightarrow \bigcup \{ f(x) \mid x \in X \} \subseteq Y \quad \text{[def. } \alpha] \]
\[ \Leftrightarrow \forall x \in X : f(x) \subseteq Y \quad \text{[def. least upper bound]} \]
\[ \Leftrightarrow X \subseteq \{ x \mid f(x) \subseteq Y \} \quad \text{[def. } \subseteq] \]
\[ \Leftrightarrow X \subseteq \gamma(Y) \quad \text{[by defining } \gamma(Y) \triangleq \{ x \mid f(x) \subseteq Y \} \} \]

Let \( \langle L, \sqsubseteq, \bot, \sqcup, \sqcap \rangle \) be a complete lattice. Let \( f \in L \mapsto L \) be an increasing function of \( L \) into \( L \). A dual narrowing on \( L \) is \( \Delta \subseteq L \times L \mapsto L \) such that

- \( \forall x, y \in L : (x \sqsubseteq y) \Rightarrow (x \sqsubseteq x \tilde{\Delta} y \sqsubseteq y) \), and \( (1) \)
- For any sequence \( \langle y_n, n \in \mathbb{N} \rangle \), \( \forall S \in L : \) the sequence \( x_0 \triangleq \bot, \ldots, \) \( (2) \)
  \[ x_{n+1} \triangleq y_n \tilde{\Delta} S \) is ultimately stationary (that is \( \exists \ell : \forall n \geq \ell : x_n = x_\ell \) )

Define the iteration for \( f \) and \( S \) with dual narrowing \( \tilde{\Delta} \) to be the sequence \( \langle f^n, n \in \mathbb{N} \rangle \) such that \( f^0 = \bot, f^{n+1} = f(f^n) \tilde{\Delta} S \) if \( f(f^n) \sqsubseteq S \) and otherwise \( f^{n+1} = f^n \) when \( f(f^n) \not\sqsubseteq S \).

**Question 4** Prove that the iteration for \( f \) and \( S \) with dual narrowing \( \tilde{\Delta} \) is ultimately stationary.

**Answer 4** Either \( \exists \ell \in \mathbb{N} : f(f^\ell) \not\sqsubseteq S \) and then, by definition of the iterates and recurrence, \( \forall n \geq \ell : f^n = f^\ell \) so that \( \langle f^n, n \in \mathbb{N} \rangle \) is ultimately stationary.

Else \( \forall \ell \in \mathbb{N} : f(f^\ell) \subseteq S \), and then by choosing \( \forall n \in \mathbb{N} : x_n = f^n \) and \( y_n \triangleq f(f^n) \), the condition (2) in the definition of the dual narrowing implies that \( \langle f^n, n \in \mathbb{N} \rangle \) is ultimately stationary.

**Question 5** Let the iteration for \( f \) and \( S \) with dual narrowing \( \tilde{\Delta} \) be ultimately stationary at rank \( \ell \in \mathbb{N} \). Prove that if \( f(f^\ell) \subseteq f^\ell \) then \( \text{lfp}^\bowtie f \subseteq S \).

**Answer 5** Observe that \( f^0 = \bot \subseteq S \) for the basis and if \( f^n \subseteq S \) then either \( f(f^n) \not\subseteq S \) in which case \( f^{n+1} = f^n \subseteq S \) or \( f(f^n) \subseteq S \) and \( f(f^n) \subseteq f^{n+1} \) \( \triangleq f(f^n) \subseteq S \) by (1) proving \( \forall n \in \mathbb{N} : f^n \subseteq S \) by recurrence. By Tarski’s theorem, \( \text{lfp}^\bowtie f = \bigcap \{ x \mid f(x) \subseteq x \} \) and so \( \text{lfp}^\bowtie f \subseteq f^\ell \) since \( f^\ell \in \{ x \mid f(x) \subseteq x \} \) and definition of a greatest lower bound. By transitivity, we conclude that \( \text{lfp}^\bowtie f \subseteq S \).