Geometric Inclusion Orders:

A New Direction in Ramsey Theory?

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Notation and Terminology

Definition. *An inclusion representation of a poset P is an assignment of a set S_x to each $x \in P$ so that*

$$x \leq y \text{ in } P \text{ if and only if } S_x \subseteq S_y.$$

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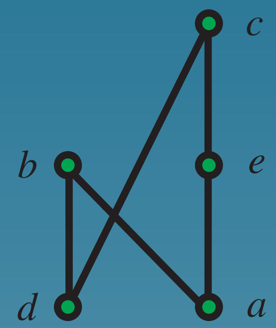
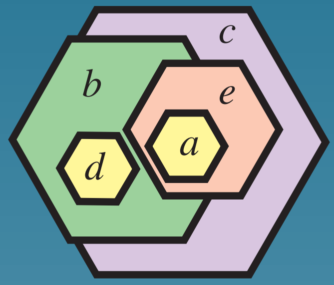
Definition. *When $\mathcal{F}$ is a family of sets and P is a poset having an inclusion representation using sets from $\mathcal{F}$, we call P an $\mathcal{F}$ inclusion order, or just an $\mathcal{F}$ order.*

Geometric Inclusion Orders





Circle Orders

Definition. *A poset P is a circle order if it has an inclusion representation using circles in the plane $\mathbb{R}^2$.*

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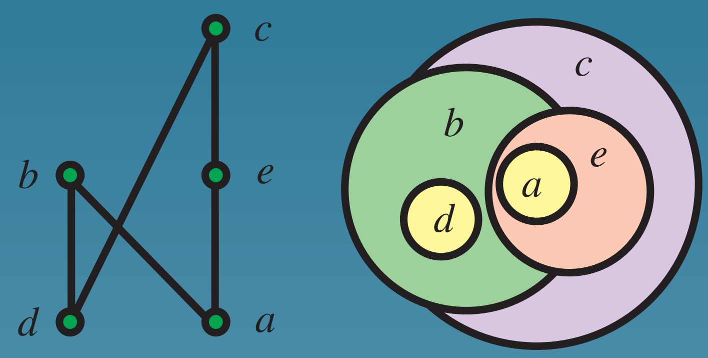

Dimension of Posets

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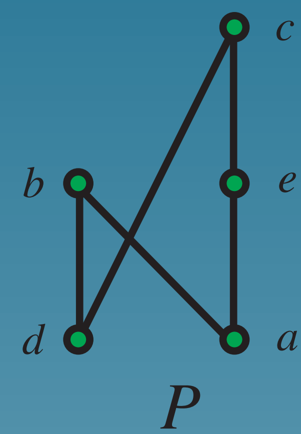


L_1	L_2
c	b
e	c
b	d
a	e
d	a

$$\dim(P) = 2$$

Alternate Definion of Dimension

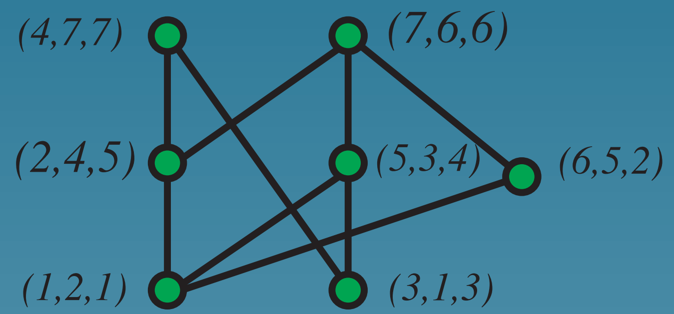
Definition. $\dim(P)$ *is also the least* t *so that* P *is isomorphic to a subposet of* $\mathbb{R}^d$ *equipped with the product ordering:*

$$(a_1, a_2, \ldots, a_t) \leq (b_1, b_2, \ldots, b_t)$$

if and only if

$$a_i \leq b_i \text{ for } i = 1, 2, \ldots, t.$$

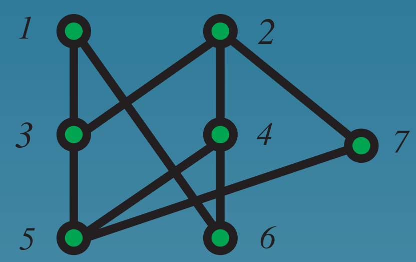
A 3-dimensional poset

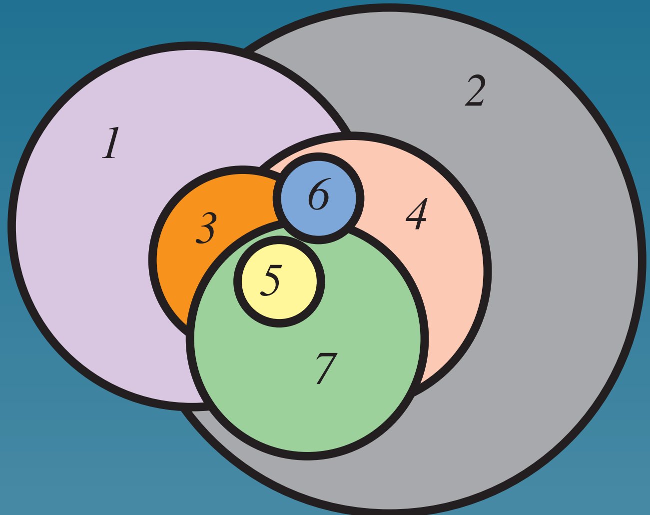


P

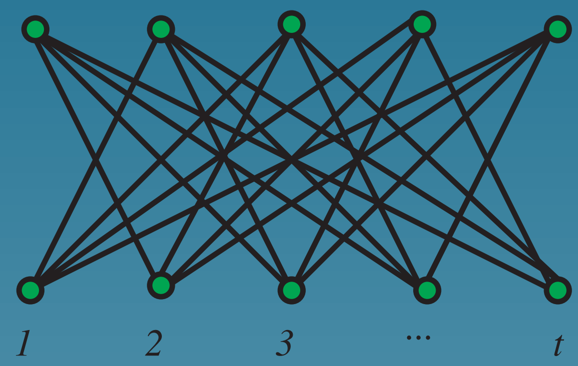
$$\dim(P) = 3$$

Some 3-dimensional Posets are Circle Orders





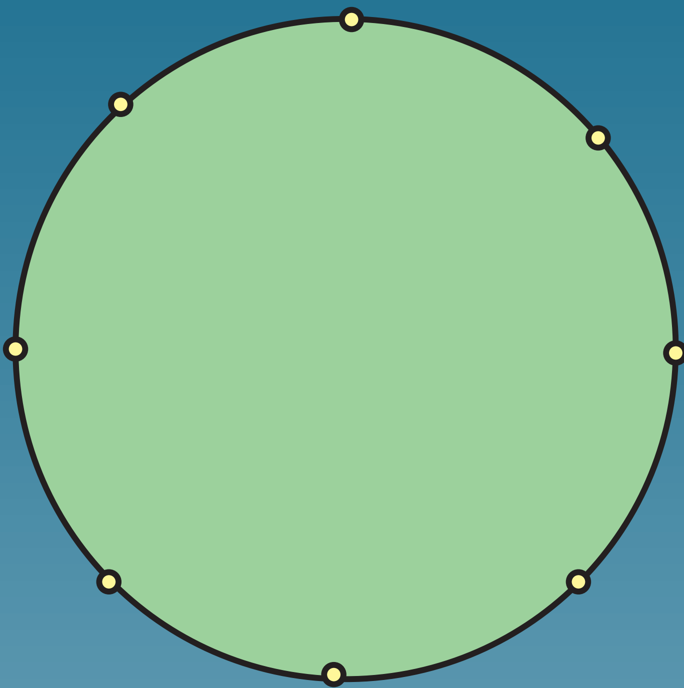
The Standard Examples



$$S_t$$

$$\dim(S_t) = t$$

The Standard Examples are Circle Orders



Sphere Orders

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Remark (?). *Every poset is a sphere order.*

Incidence Posets of Graphs

Definition. *Let G be a graph. The incidence poset of G, denoted P_G, consists of the vertices and edges of G ordered by inclusion.*

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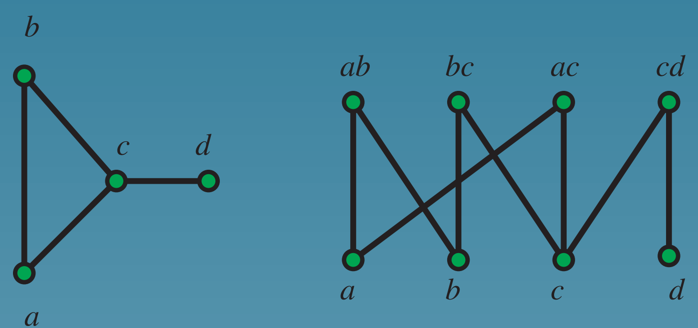

G $\qquad$ P_G

Planar Graphs

Theorem. [Schnyder, 1989; Scheinerman, 1991]
Let G be a finite graph and let P_G be its incidence poset. Then the following statements are equivalent:

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1. *G is a planar graph.*
2. *The dimension of P_G is at most 3.*
3. *P_G is a circle order.*

Fundamental Question for Circle Orders

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Theorem. *For every $t \geq 3$, the standard example S_t is a t-dimensional poset which is also a circle order. On the other hand, almost all 4-dimensional posets are NOT circle orders.*

The Answer Should be YES!!

Remark. *For every* $n \geq 3$, *every finite* 3*-dimensional poset has an inclusion representation using regular* n*-gons.*

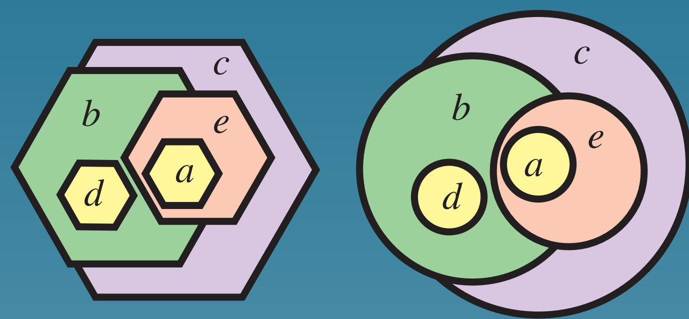
The Answer Should be YES!!

Remark. *For every $n \geq 3$, every finite 3-dimensional poset has an inclusion representation using regular n-gons.*

Remark. *Every finite 3-dimensional poset has an inclusion representation using ellipses.*

When n is LARGE??

Doesn't a regular n-gon turn into a circle as n increases?



The Answer Should be **NO!!**

Theorem. [Scheinerman and Wierman, 1988] *The countably infinite poset* $\mathbb{Z}^3$ *is not a circle order.*

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Theorem. [Fon-der-Flaass, 1993] *The countably infinite poset $\mathbf{2} \times \mathbf{3} \times \mathbb{N}$ is not a sphere order.*

A Strange Conjecture on Sphere Orders

Theorem. [Maehara, 1984] *If G is a graph, then there exists an integer k for which G is the intersection graph of a family of spheres in $\mathbb{R}^k$.*

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Conjecture. [Brightwell and Winkler, 1989]
To the contrary, there exists a finite poset which is not a sphere order.

The Surprising (?) Answer!!!

Theorem. [Felsner,Fishburn and Trotter, 1997]
There exists a positive integer n_0 so that if $n > n_0$, the finite 3-dimensional poset $\mathbf{n} \times \mathbf{n} \times \mathbf{n}$ is NOT a sphere order.

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Remark. *If P is a t-dimensional poset, then P has an inclusion representation using cubes in $\mathbb{R}^{t+1}$. This result is best possible.*

Part II: Sketch of the Proof

Change Patterns for Increasing Sequences

Definition. *Let N be a fixed (large) positive integer. Then consider an increasing sequence of positive real numbers:*

$$0 < a_1 < a_2 < a_3 < a_4 < \cdots < a_n.$$

The sequence advances conservatively in magnitude (ACM) if

$$i < j \textit{ implies } a_j > N a_i.$$

The sequence is nearly constant (NC) if

$$i < j \textit{ implies } a_j < (1 + 1/N) a_i.$$

Nearly Constant Sequences

Definition. *A NC sequence advances conservatively (AC) if*

$$i < j < k \text{ implies } a_k - a_j > N(a_j - a_i).$$

An NC sequence advances agressively (AA) if

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Proposition. *For every m, there exists n_0 so that if $0 < a_1 < a_2 < a_3 < a_4 < \cdots < a_n$ is an increasing sequence of positive real numbers and $n > n_0$, then there is a subsequence of length m which is either (1) ACM; (2) AC, or (3) AA.*

Decreasing Sequences

Definition. *For decreasing sequences, the analogous terms are:*

1. *Retreating Agressively in Magnitude (RAM).*
2. *Retreating Agressively (RA).*
3. *Retreating Conservatively (RC).*

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Proposition. *For every m, there is an n_0 so that if $n > n_0$, then for any sequence of n distinct positive real numbers, there is a subsequence of length m satisfying one of the six change patterns: ACM, AC, AC, RAM, RA and RC.*

The Product Ramsey Theorem

Definition. *For positive integers n, k and t, a $\mathbf{k}^t$ grid in $\mathbf{n}^t$ is a set of the form $S_1 \times S_2 \times \ldots S_t$ where each S_i is a k-element subset of $\{0, 1, \ldots, n-1\}$.*

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Theorem. [Product Ramsey Theorem] *Given positive integers m, k, r and t, there exists an integer n_0 so that if $n \geq n_0$ and f is any map which assigns to each $\mathbf{k}^t$ grid of $\mathbf{n}^t$ a color from $\{1, 2, \ldots, r\}$, then there exists a subposet P isomorphic to $\mathbf{m}^t$ and a color $\alpha \in \{1, 2, \ldots, r\}$ so that $f(g) = \alpha$ for every $\mathbf{k}^t$ grid g from P.*

Monotonic Functions

Definition. *A function f mapping $\mathbf{n}^t$ to the positive reals is*

1. *order-preserving if $x < y$ implies $f(x) \le f(y)$.*

2. *order-reversing if $x < y$ implies $f(x) \ge f(y)$.*

3. *monotonic if f is either order preserving or order reversing.*

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Corollary. *For every m, t, there exists n_0 so that if $n > n_0$ and f is any injective function mapping $\mathbf{n}^t$ to the positive reals, then there is a subposet isomorphic to $\mathbf{m}^t$ such that the restriction of f is monotonic.*

Coordinate Domination

Definition. *Let f be an injective order preserving function mapping $\mathbf{n}^t$ to the positive reals. f is dominated by coordinate α if*

$$f(x) < f(y) \text{ whenever } x(\alpha) < y(\alpha).$$

Similarly, if f is order reversing, then we require

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Definition. *Let f be an injective order preserving function mapping $\mathbf{n}^t$ to the positive reals. f is dominated by coordinate α if*

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Theorem. [Fishburn and Graham, 1993]
For every m, t, there exists n_0 so that if $n > n_0$ and f is any injective function mapping $\mathbf{n}^t$ to the positive reals, then there is a subposet isomorphic to $\mathbf{m}^t$ and an integer α such that the restriction of f is monotonic and dominated by coordinate α.

N-Uniform Functions

Definition. *Let f be an injective function mapping $\mathbf{n}^t$ to the positive reals. f is N-uniform if*

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2. *There is a coordinate α dominating f.*

3. *There is a change label from ACM, AC, AA, RAM, RA, RC which f satisfies.*

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Theorem. [Felsner, Fishburn and Trotter, 1997]
For every m, t, N, there exists n_0 so that if $n > n_0$ and f is any injective function mapping $\mathbf{n}^t$ to the positive reals, then there is a subposet isomorphic to $\mathbf{m}^t$ and an integer α such that the restriction of f is N-uniform and dominated by coordinate α.

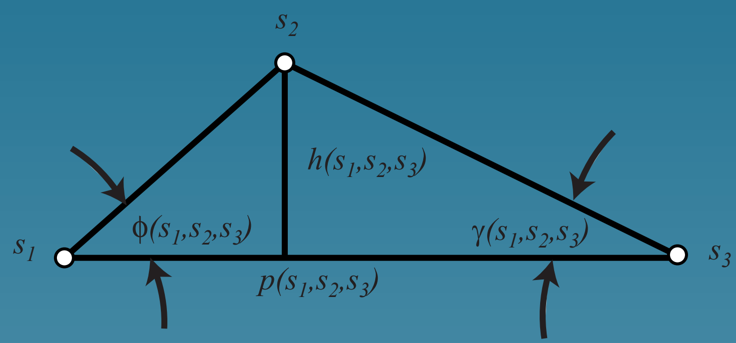
Induced Functions

Definition. *Let $\mathbf{s} \in \mathbf{k}^t$ and let A be a function which maps $\mathbf{k}^t$ grids to $\mathbb{R}$. Then each $(\mathbf{k}-\mathbf{1})^t$ grid g induces a function $A_{g,\mathbf{s}}$ defined on points from the products of chains. For each $i = 1, 2, \ldots, t$, the points from the i^{th} factor are those between the s_{i-1}^{st} and s_i^{th} point of the i^{th} factor set of g. The different functions are called types.*

Uniformizing Induced Functions

Theorem. [Felsner,Fishburn and Trotter, 1997]
For every m, k, t, N, there exists n_0 so that if $n > n_0$ and A is any injective function mapping the k^t grids of $\mathbf{n}^t$ to the positive reals, then there is a subposet Q isomorphic to $\mathbf{m}^t$ and an a collection of change patterns, one for each of the k^t functions induced by a $(k-1)^t$ grid, so that all induced functions on Q are N-uniform and satisfy a change pattern which depends only on the type—and not on the grid.

Assume $\mathrm{n} \times \mathrm{n} \times \mathrm{n}$ is a Sphere Order



Functions on Grids

Definition. *With each 3^3 grid g, we associate a 3-element chain $x < y < z$. We then set:*

1. $A(g) = \phi(x, y, z)$.
2. $B(g) = h(x, y, z)$.
3. $C(g) = h(x, y, z)\phi(x, y, z)/2$.

Basic Notation

Definition. *1. $r(x)$ is the radius of x.*

2. $\rho(x, y)$ is the distance between $c(x)$ and $c(y)$. $c(x)c(y)$ and $c(x)c(z)$.

3. More stuff

Remark. *We consider the following induced functions:*

1. $\Phi(y) = \phi(x, y, z)$.

2. $\Theta(z) = \phi(x, y, z)$.

3. $H(y) = h(x, y, z)$.

4. $K(x) = h(x, y, z)$.

5. $G(y) = h(x, y, z)\phi(x, y, z)/2$.

Some Details of the Proof

Remark. *We may assume that the radius function is ACM and dominated by coordinate 1. If it is AA, we invert and use the fact that the dual of a sphere order is a sphere order. With this change, the radius function is AC.*

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If the radius function is AC, then we subtract an appropriate quantity to make it ACM.

Completing the Proof

Remark. *The remainder of the argument is by case analysis, depending on the change labels for the induced functions determined by* Φ, Θ, H, K *and* G. *Surprisingly, we are able to argue that there are essentially only three cases. Furthermores, two of these three cases are dual—using a weak form of the triangle inequality.*

Completing the Proof

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For example, we show:

1. *The function Φ cannot be ACM.*

2. *The function H cannot be RAM.*

3. *If Φ is NC, then H is ACM and dominated by coordinate 1.*

4. *If H is NC, then Φ is RAM and dominated by coordinate 1.*

The Geometric Part of the Proof

Definition. *For distinct points x and y from $\mathbf{n}^3$,*

$$\mathrm{Gap}(x, y) = r(y) - r(x) - \rho(x, y).$$

Remark. *When $x < y$, $\mathrm{Gap}(x, y) > 0$, and when x is incomparable to y, $\mathrm{Gap}(x, y) < 0$.*

Definition. *For three distinct points x, y and z, let*

$$\Delta(x, y, z) = \rho(x, y) + \rho(y, z) - \rho(x, z).$$

Remark. $\Delta(x, y, z) \geq 0$, *and* $\Delta(x, y, z) > 0$ *when the centers are not collinear.*

The Geometric Part of the Proof (2)

Consider a 2-element chain $x < z$ and a point v incomparable to both. Then

$$\begin{aligned} r(z) - r(x) &= \big(r(v) - r(x)\big) + \big(r(z) - r(v)\big) \\ &< \rho(x, v) + \rho(v, z), \end{aligned}$$

so that

$$\operatorname{Gap}(x, z) < \Delta(x, v, z).$$

Since this bound holds for any point incomparable to both x and z, we may consider several candidate points and take the best bound they produce. As a result, we have an upper bound on $\operatorname{Gap}(x, z)$.

The Geometric Part of the Proof (3)

Let $C = \{x = u_1 < u_2 < \cdots < u_{2k+1} = z\}$ be a chain. Then

$$\begin{aligned}
r(z) - r(x) &= r(u_{2k+1}) - r(u_1) \\
&= \sum_{i=1}^{2k} \big[r(u_{i+1}) - r(u_i)\big] \\
&> \sum_{i=1}^{2k} \rho(u_{i+1}, u_i) \\
&= \sum_{i=1}^{k} \big[\rho(u_{2i+1}, u_{2i-1}) + \Delta(u_{2i-1}, u_{2i}, u_{2i+1})\big] \\
&\geq \rho(u_1, u_{2k+1}) + \sum_{i=1}^{k} \Delta(u_{2i-1}, u_{2i}, u_{2i+1}). \\
&= \rho(x, z) + \sum_{i=1}^{k} \Delta(u_{2i-1}, u_{2i}, u_{2i+1}).
\end{aligned}$$

The Geometric Part of the Proof (4)

Setting

$$\Delta(x, C, z) = \sum_{i=1}^{k} \Delta(u_{2i-1}, u_{2i}, u_{2i+1}),$$

we conclude that

$$\mathrm{Gap}(x, z) > \Delta(x, C, z).$$

Now we have a lower bound on $\mathrm{Gap}(x, z)$.

We obtain a contradiction by carefully choosing the point v and the chain C so that

$$\Delta(x, v, z) < \Delta(x, C, z).$$

The Geometric Part of the Proof (5)

As indicated previously, there are three cases:

Case 1. Φ is RAM; H is ACM.

Case 2. Φ is NC; H is ACM.

Case 3. H is NC; Φ is RAM.

Furthermore, Case 2 and Case 3 are dual.