LINKLESS EMBEDDINGS OF GRAPHS IN 3-SPACE

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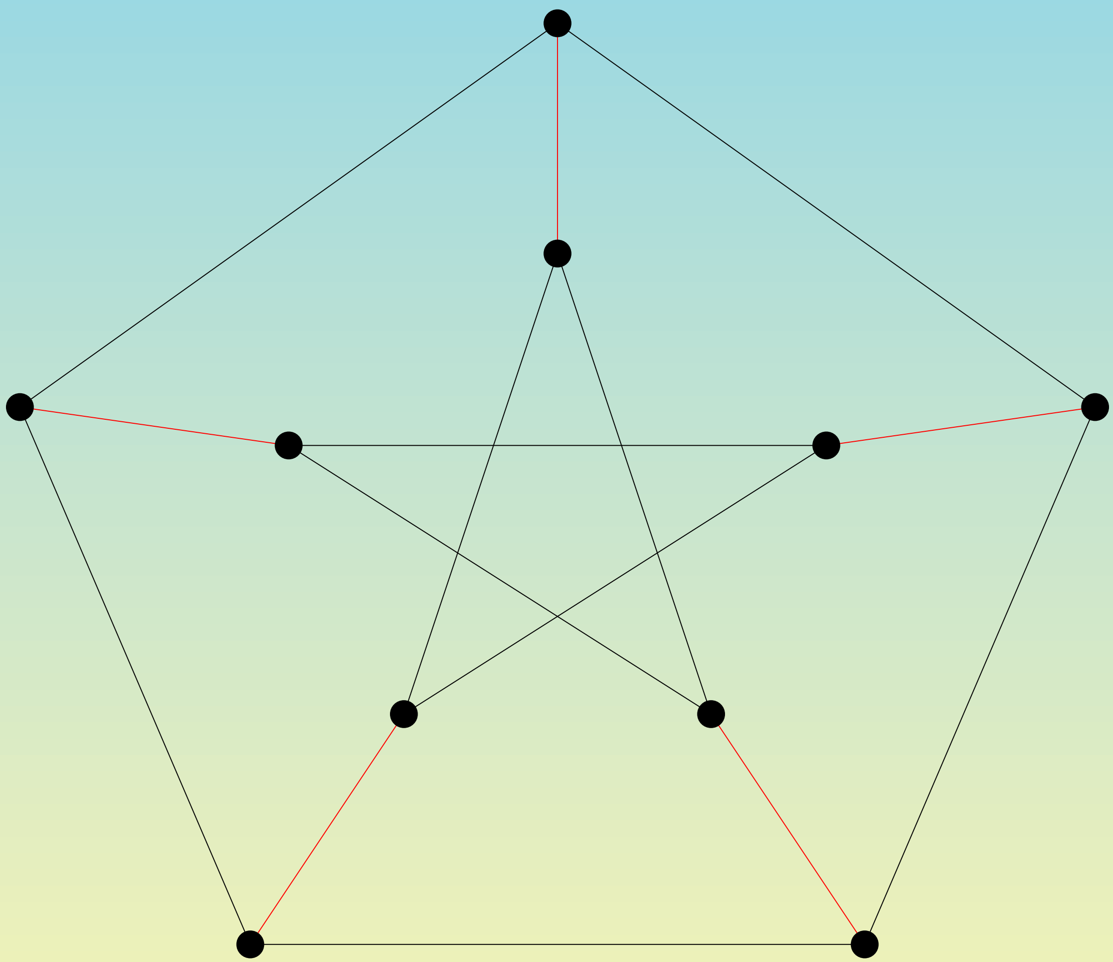
joint work with

N. Robertson and P. D. Seymour

Graphs are finite, undirected, may have loops and multiple edges. H is a minor of G if H can be obtained from a subgraph of G by contracting edges.



KNOT THEORY

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THEOREM (Papakyriakopolous) A simple closed curve in $\mathbb{R}^3$ is unknotted $\Leftrightarrow$ its complement has free fundamental group.

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- $= 1$ for some embedding

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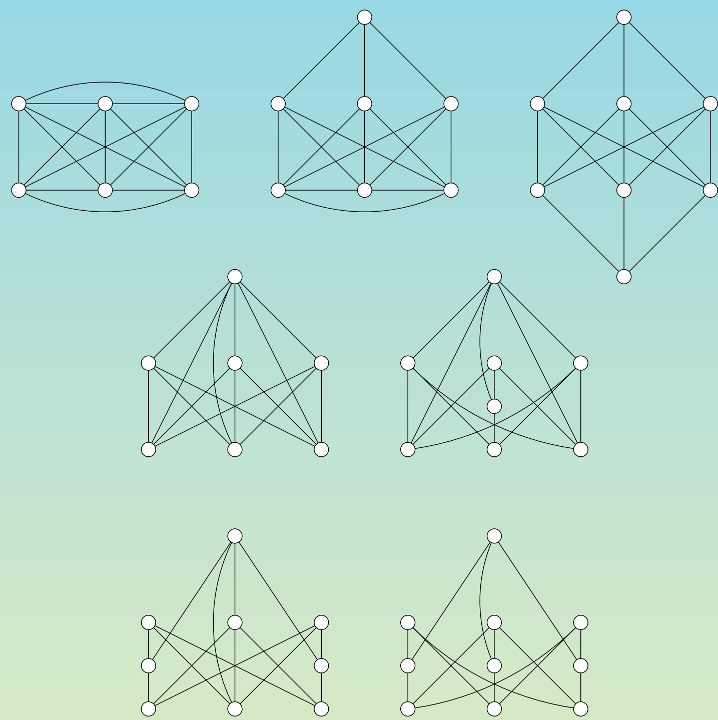

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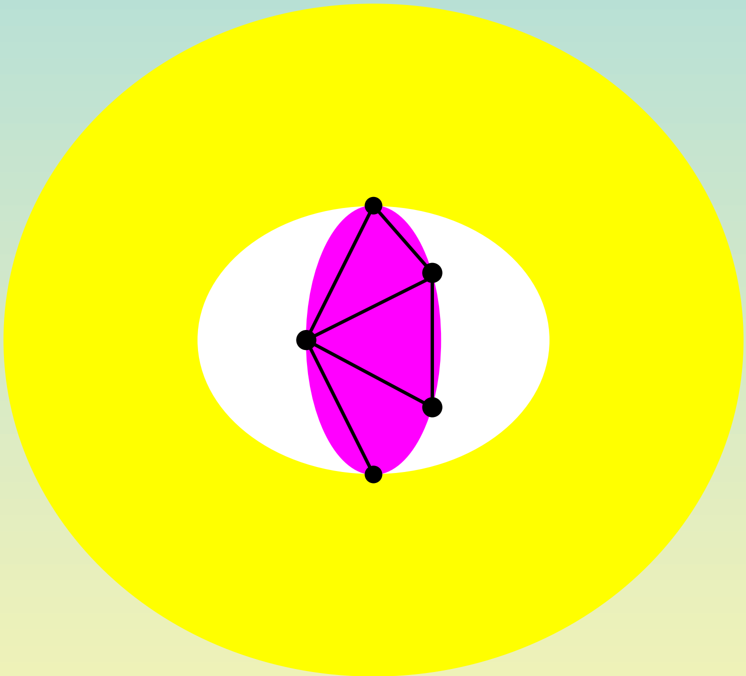

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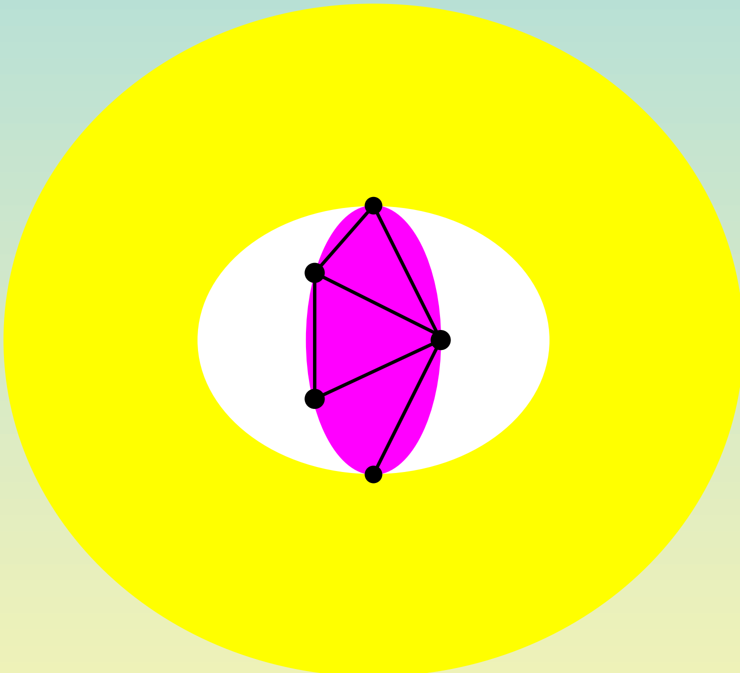

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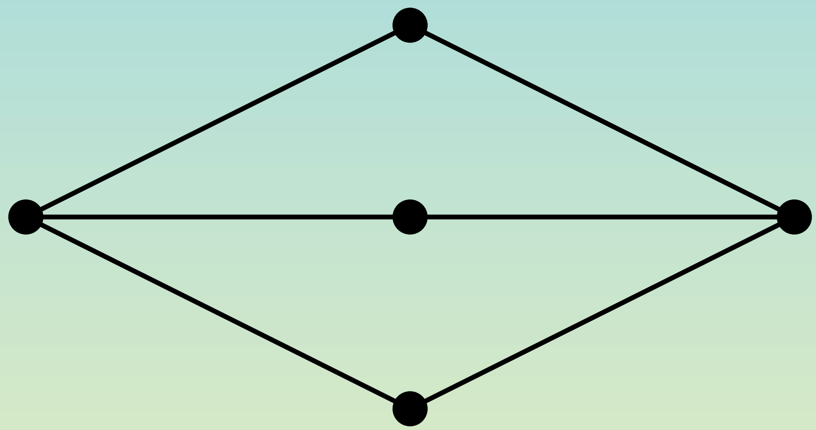

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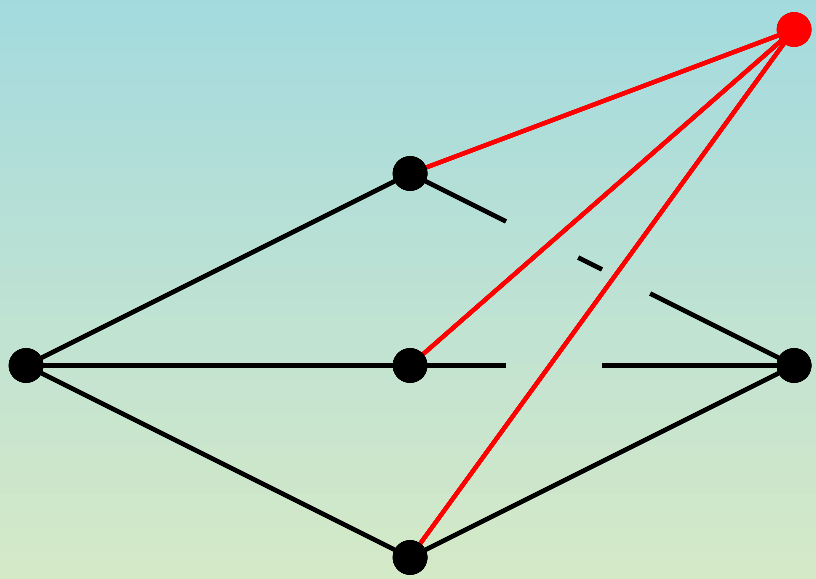

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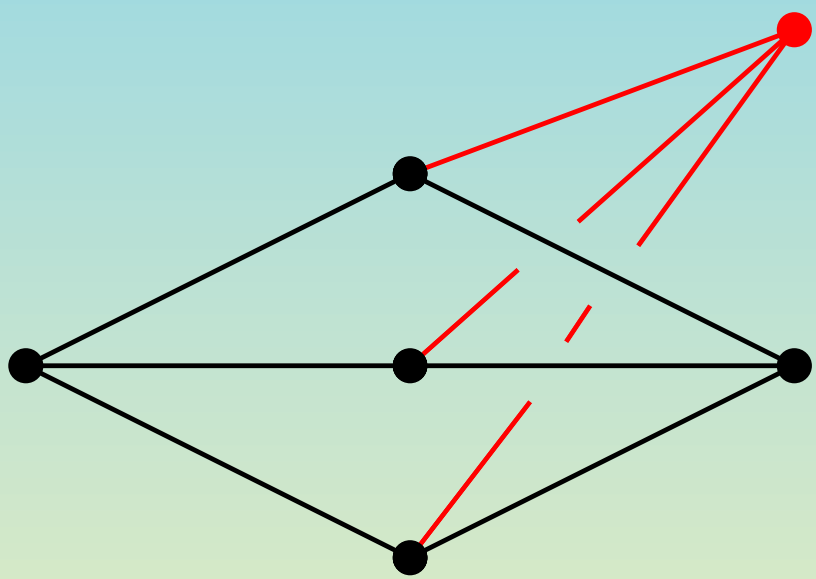

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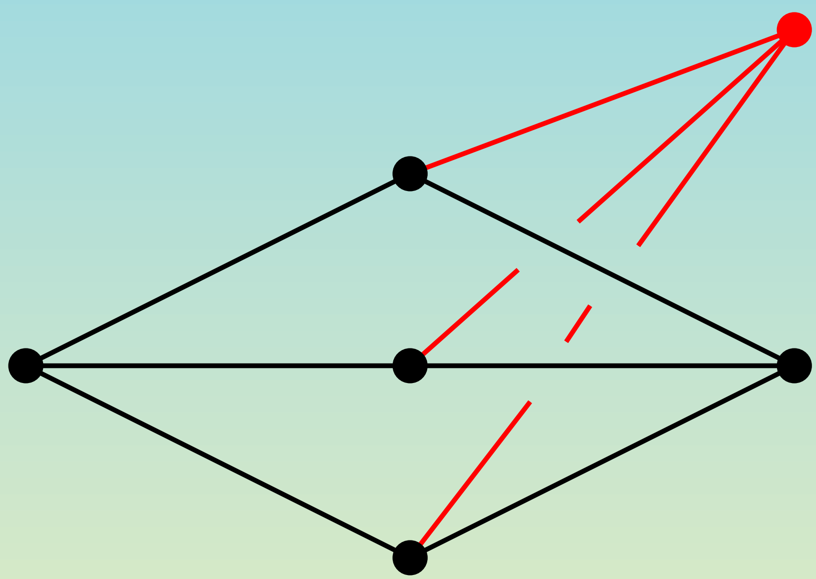

COROLLARY A graph has a unique flat embedding $\Leftrightarrow$ it is planar.

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LEMMA In a 4-connected graph G, every two $K_{3,3}$ minors "communicate".

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MAIN THEOREM

G has no minor isomorphic to a member of the Petersen family $\Rightarrow$ G has a flat embedding.

OUTLINE OF PROOF Take a minor-minimal counterexample, G, WMA no triangles. It can be shown G is "internally 5-connected." Take edges $e = uv$, f such that $G/f/e$, $G/f\backslash e$ are "Kuratowski connected."

Let ϕ_1 be a flat embedding of $G\backslash e$.

Let ϕ_2 be a flat embedding of G/e.

Let ϕ_3 be a flat embedding of G/f. WMA

$$\phi_1/f \simeq \phi_3\backslash e$$
$$\phi_2/f \simeq \phi_3/e$$

It can be shown that the uncontraction of f is the same in both of these embeddings. Let ϕ be obtained from ϕ_3 by doing this uncontraction. Then $\phi\backslash e \simeq \phi_1$, $\phi/e \simeq \phi_2$, and so ϕ is flat.

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JORGENSEN'S CONJECTURE Every 6-connected graph with no K_6-minor is apex (=planar + one vertex).

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NOTE No explicit algorithm is known.