You are expected to solve all the problems, but for grading purposes, submit written solutions to any 4 of the problems. Due on Mon, March 16th. Collaboration is allowed; please mention your collaborators.

1. Show that $\Sigma_k = \text{NP}^{\text{SAT}_{k-1}}$.

   We define $L \in \Sigma_k$ if and only if there is a deterministic polynomial time verifier $V$ such that
   
   $$x \in L \iff \exists y_1 \forall y_2 \cdots Q_k y_k \ V(x, y_1, \ldots, y_k) = 1$$
   
   where length of $y_1, y_2, \ldots, y_k$ is bounded by a polynomial in $|x|$.

2. • Show that $\text{PSPACE} = \text{NP}^{\text{PSPACE}} = \text{PSPACE}$

   • Show that if $\text{PH} = \text{PSPACE}$, then $\text{PH}$ collapses to some finite level.

   • Can $\text{PH}$ have a complete problem (complete under polynomial time reductions)?

3. **(DP-completeness)** This problem studies the class DP (D stands for difference). A language $L \in DP$ if and only if there are languages $B \in \text{NP}$ and $C \in \text{coNP}$ so that $L = B \cap C$.

   • The problem SAT-UNSAT is defined as follows: Given Boolean formulae $\phi, \psi$, decide if $\phi$ is satisfiable And $\psi$ is unsatisfiable. Show that this problem is DP-complete (under polynomial time reductions).

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A graph $G$ is in HC-CRITICAL if $G$ is not Hamiltonian but adding any edge to $G$ will make it Hamiltonian. Show that HC-CRITICAL is in DP.

4. • Show that $\text{NP}^{\text{BPP}} \subseteq \text{BPP}^{\text{NP}}$ (Hint: First show that a language in $\text{NP}^{\text{BPP}}$ is accepted by a polytime NTM that makes a single query to a BPP oracle and that too at the end).

• Show that if $\text{NP} \subseteq \text{BPP}$, then PH collapses to $\text{BPP}$.

5. (NEXP-completeness) Define $\text{NEXP} = \cup_{k=1,2,...} \text{NTIME}(2^{n^k})$.

Show that the following problem is NEXP-complete: Given $<M,x,n>$, consisting of description of a NTM $M$, input $x$ and an integer $n$ in binary, does $M$ have an accepting computation on $x$ in $n$ steps.

6. A circuit $C$ is called an implicit representation of another circuit $C^*$ if $C$ takes as input a binary integer $i$ such that $n+1 \leq i \leq N$, and outputs a triple $(\text{TYPE},j,k)$ where

• Input to $C^*$ is an $n$-bit string $x_1x_2\ldots x_n$.

• $\text{TYPE} \in \{\text{AND}, \text{OR}, \text{NOT}\}$ indicates the type of $i^{th}$ gate in circuit $C^*$.

• $1 \leq j,k \leq N$.

• The input of the $i^{th}$ gate in $C^*$ is the output of the $j^{th}$ and $k^{th}$ gates of $C^*$ (if TYPE= NOT, then $k$ is ignored. If $1 \leq j,k \leq n$, then the $j^{th}$ or $k^{th}$ gate is taken to be an input bit $x_i$).

• The $N^{th}$ gate in $C^*$ is its output gate.

Note that we could have $N = 2^n$, the circuit $C$ could be of size $\text{poly}(\log N) = \text{poly}(n)$ and still implicitly represent a circuit $C^*$ of size $N$ (in short, a circuit can implicitly represent another circuit of size exponential in its own size).

Let IMPLICIT CIRCUIT-SAT be the following problem: Given a circuit $C$ that is an implicit representation of circuit $C^*$, decide if $C^*$ is satisfiable. Show that this problem is NEXP-complete (Hint: Use the regular structure of the circuit produced in Cook’s reduction).

7. • Show that $\text{NP}^{\text{NP} \cap \text{coNP}} = \text{NP}$.

• Generalize this to $\text{NP}^{\Sigma_1 \cap \Pi_k} = \Sigma_k$. 

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8. The problem Graph Consistency (GC) asks, for two given sets A and B of graphs, whether there exists a graph G such that every graph $g \in A$ is isomorphic to a (not necessarily induced) subgraph of G but each graph $h \in B$ is not isomorphic to any subgraph of G. Show that GC is in $\Sigma_2$. (Optional: show that it is $\Sigma_2$-complete).

9. Show that if $\Sigma_k = \Pi_k$ for some $k$, then $PH = \Sigma_k$. 