The Problem:

“Richard Bronowitz offers this radar detection problem. Assume that a radar has a detection thresh-
old requiring at least nine successful pulse returns out of 10 successive pulses. Furthermore, once
an object is is detected, it remains detected – i.e., there are no lost contacts. The probability that
a pulse is successfully detected is \( p \), and pulse results are independent. What is the probability of
detection given \( N \) pulses?”

Overview. This problem may be re-characterized and generalized:

Flip a biased coin with \( \Pr\{\text{head}\} = p \) until attainment of the goal “\( K \) heads occur among the final
\( M \) flips for the first time at the \( N \)th flip”. Determine the cumulative probability distribution for
random variable \( N \).

A Solution. This situation lends itself to modeling as an Absorbing Markov Chain\(^*\). Definitions of
states and transitions are:

State: Strings of length \( M \) of 0’s and 1’s, a 1 representing a head flip. Such strings containing \( K \) or
more 1’s are aggregated to constitute the one absorbing state \( G \), representing the attainment of the goal.
All other strings are individual transient states.

Transition: A flip of the coin. If \( b_1 b_2 \ldots b_M \) is a current transient state (\( b_i \) is a binary digit), then transition probabilities are

\[
\begin{align*}
\Pr\{b_2 \ldots b_M 1 \mid b_1 b_2 \ldots b_M\} &= p, \\
\Pr\{b_2 \ldots b_M 0 \mid b_1 b_2 \ldots b_M\} &= 1 - p = q, \quad \text{and} \\
\Pr\{G \mid G\} &= 1.
\end{align*}
\]

Very Small Example. With parameter values \( K = 2 \) and \( M = 3 \), the one-step transition matrix \( T \) is

\[
T = \begin{pmatrix}
G & 000 & 001 & 010 & 100 \\
000 & 1 & 0 & 0 & 0 \\
001 & 0 & q & p & 0 \\
010 & p & 0 & 0 & q \\
100 & 0 & q & p & 0
\end{pmatrix}
\]

Define state vector \( S_n \), the condition of the process at time \( n \) (after \( n \) flips), by

\[
S_n = ( G(n), \ s_n(1), \ s_n(2), \ s_n(3), \ s_n(4) ),
\]

in which \( G(n) = \Pr\{N \leq n\} \) and \( s_n(i) = \Pr\{\text{at time } n \text{ the process is in transient state } i\} \).

With initial state vector \( S_0 = (0,1,0,0,0,0) \), successive state vectors are determined by recursion:

\[
S_n = S_{n-1} T, \quad n = 1, 2, \ldots \quad (R)
\]

The sequence

\[
\{G(n) \mid n = 0, 1, 2, \ldots\}
\]

is the cumulative probability distribution for random variable \( N \).

\(^*\) See, for example, the chapter on Markov Chains in *Introduction to Operations Research* by Hillier &
With a CAS, symbolic values for \( G(n) \) can be obtained. Numerical values for \( G(n) \) can be obtained for a desired \( p \). Graphs of \( G(n) \) for \( p = 0.1 \) to \( p = 0.9 \) are given for the Very Small Example:

![Graphs of G(n) for p = 0.1 to p = 0.9](image)

**Large Values for Parameters.** For large values of \( K \) and \( M \) the approach above is impractical. In general, matrix \( T \) would have \( 2^M - \sum_{i=K}^{M-1} \binom{M}{i} \) rows and columns. For values \( K = 9 \) and \( M = 10 \) in the original problem statement, the size of matrix \( T \) is \( 1014 \times 1014 \). Because matrix \( T \) is sparse, with just two non-zero entries per row, an alternative is to replace the recursion (R) with explicit equations. For the Very Small Example and \( n = 1, 2, \ldots \), these equations are

\[
G(n) = G(n-1) + p \cdot [s_{n-1}(2) + s_{n-1}(3)],
\]
\[
s_n(1) = q \cdot [s_{n-1}(1) + s_{n-1}(4)],
\]
\[
s_n(2) = p \cdot [s_{n-1}(1) + s_{n-1}(4)],
\]
\[
s_n(3) = q \cdot s_{n-1}(2), \quad \text{and}
\]
\[
s_n(4) = q \cdot s_{n-1}(3).
\]
As the coin is flipped the Markov Chain transitions from state to state. Denote the transient states by \( B_1, B_2, \ldots \) The possible transitions are depicted here:

**State Transition Diagrams (Transient States)**

Next Flip a Tail (\( q \))          Next Flip a Head (\( p \))

\[
\begin{align*}
0b_1 \ldots b_{M-1} \quad &\quad 0b_1 \ldots b_{M-1} \\
\downarrow \quad &\quad \downarrow \\
b_1 b_2 \ldots b_{M-1} \quad &\quad b_1 b_2 \ldots b_{M-1} \\
\downarrow \quad &\quad \downarrow \\
1b_1 \ldots b_{M-1} \quad &\quad 1b_1 \ldots b_{M-1}
\end{align*}
\]

Those states which contain \( K \) (or more) 1’s are aggregated into a single absorbing state \( G \). If state \( B_{j_1} \) contains \( K \) 1’s, then \( B_{j_1} \equiv G \). In general, each state has two states preceding.

As earlier, \( s_n(i) \) is the probability that the system is in transient state \( B_i \) at time \( n \). Then equations replacing recursion (R) are of the form:

\[
\begin{align*}
s_n(j_0) &= q \cdot [s_{n-1}(i_0) + s_{n-1}(i_1)], \\
s_n(j_1) &= p \cdot [s_{n-1}(i_0) + s_{n-1}(i_1)].
\end{align*}
\]

**Numerical Examples.** Values for \( G(n) \), \( n = 0, 1, \ldots \), including initial parameter values \( K = 9 \) and \( M = 10 \), are given in the following figures. Variations in the three parameters \( K \), \( M \), and \( p \) are explored. In each plot, value \( n \) extends up the first flip on which \( G(n) \geq 0.998 \) (with a maximum 200 flips). For emphasis, figures displaying parameter variations have origins at \( (0, 0.75) \).

**Cumulative Probability Distribution \( G(n) \) with \( K = 9 \) and \( M = 10 \) (\( p = 0.9 \))**
Cumulative Probability Distributions $G(n)$ with $M = 10$ and $p = 0.9$ ($K = 6[1][9]$)