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Problem S/O 3. Show that the solutions of $e^z = \frac{z-1}{z+1}$ are on the imaginary axis.

Solution

Let x and y be the real and imaginary parts of z, and $z = x + iy$ be a root of :

$$f(z) = e^z - \frac{z-1}{z+1} \tag{1}$$

The root necessarily satisfies the weaker condition: $|e^z| = \left|\frac{z-1}{z+1}\right|$ which is equivalent to:

$$e^{2x} = \frac{(x-1)^2+y^2}{(x+1)^2+y^2} \tag{2}$$

For $x > 0$, $(x-1)^2 < (x+1)^2$ therefore $\frac{(x-1)^2+y^2}{(x+1)^2+y^2} < 1$. On the other hand, $e^{2x} > 1$ for $x > 0$. The direction of each inequality above reverses for $x < 0$. Let's summarize these results in a table:

$x < 0$	$x > 0$
$\frac{(x-1)^2+y^2}{(x+1)^2+y^2} > 1$	$\frac{(x-1)^2+y^2}{(x+1)^2+y^2} < 1$
$e^{2x} < 1$	$e^{2x} > 1$
$e^{2x} \neq \frac{(x-1)^2+y^2}{(x+1)^2+y^2}$	$e^{2x} \neq \frac{(x-1)^2+y^2}{(x+1)^2+y^2}$

Table 1. Eq. (2) cannot be satisfied for $x \neq 0$.

Since Eq (2) is a necessary condition, any solution of $e^z = \frac{z-1}{z+1}$ must have zero real-part.

This completes the proof of the original problem. Now we can have even more fun finding the roots. Since any root of $f(z)$ is on the imaginary axis, we have:

$$e^{iy} = \frac{iy-1}{iy+1} \tag{3}$$

We observe that if iy is a root of $f(z)$, so is $-iy$:

$$e^{iy} = \frac{iy-1}{iy+1} \;\Rightarrow\; e^{-iy} = \frac{iy+1}{iy-1} = \frac{-iy-1}{-iy+1}$$

Since $z = 0$ is not a root, and the set of roots have reflection symmetry, we search the roots on the positive imaginary axis. We take the logarithm of both sides of Eq (3):

$$e^{iy} = \frac{iy-1}{iy+1} = \frac{(y+i)^2}{y^2+1} = \exp\left(2i \text{ atan } \frac{1}{y}\right) \tag{3b}$$

$$y = 2n\pi + 2 \text{ atan } \frac{1}{y} \tag{4}$$

$$g_n(y) \equiv 2n\pi + 2 \text{ atan } \frac{1}{y} \tag{5}$$

n is a non-negative integer. This transcendental equation is solved by the fixed-point iteration (W. Rudin, *Principles of Mathematical Analysis*, p. 220, 3rd Ed. McGraw Hill, 1976):

$$y \leftarrow g_n(y) \tag{6}$$

The left-arrow denotes assignment. We iterate this assignment until convergence. We find a root y_n of Eq (4) for each nonnegative integer n. The roots of the original equation (1) are $z_n = \pm i y_n;\ \ n = 0,1,2,\ldots$:

n	z_n
0	±1.30654237418881 *i*
1	±6.58462004256417 *i*
2	±12.72324078413133 *i*
3	±18.95497141084159 *i*
4	±25.21202688855082 *i*
5	±31.47943871200974 *i*

Table 2. First five pairs of infinitely many solutions of $e^z = (z-1)/(z+1)$ to 14 decimal places.

Fixed-point iteration applies because $g_n(y)$ is a contraction map (that is, $|g_n'(y)| < 1$) for $y > 1$; and Eq (4) can have no solutions for $0 < y \leq 1$. The latter statement is true because $g_n(y) \geq \frac{\pi}{2}$ for $0 < y \leq 1$.

The roots of $f(z)$ asymptote to: $z_n \cong \pm\left(2\pi i n + \frac{i}{\pi n}\right)$ as $|n| \longrightarrow \infty$. The asymptote is obtained by solving the quadratic equation:

$$y \cong 2n\pi + \frac{2}{y} \quad \text{as } n \longrightarrow \infty,$$

which is the large y approximation to Eq (4).

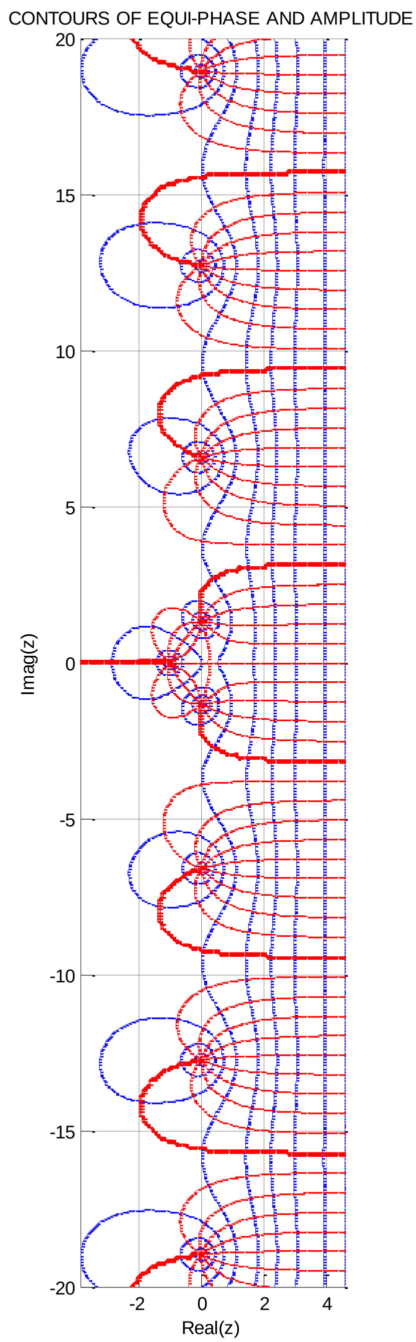
CONTOURS OF EQUI-PHASE AND AMPLITUDE
Imag(z)
Real(z)
20
15
10
5
0
-5
-10
-15
-20
-2
0
2
4

Fig. 1

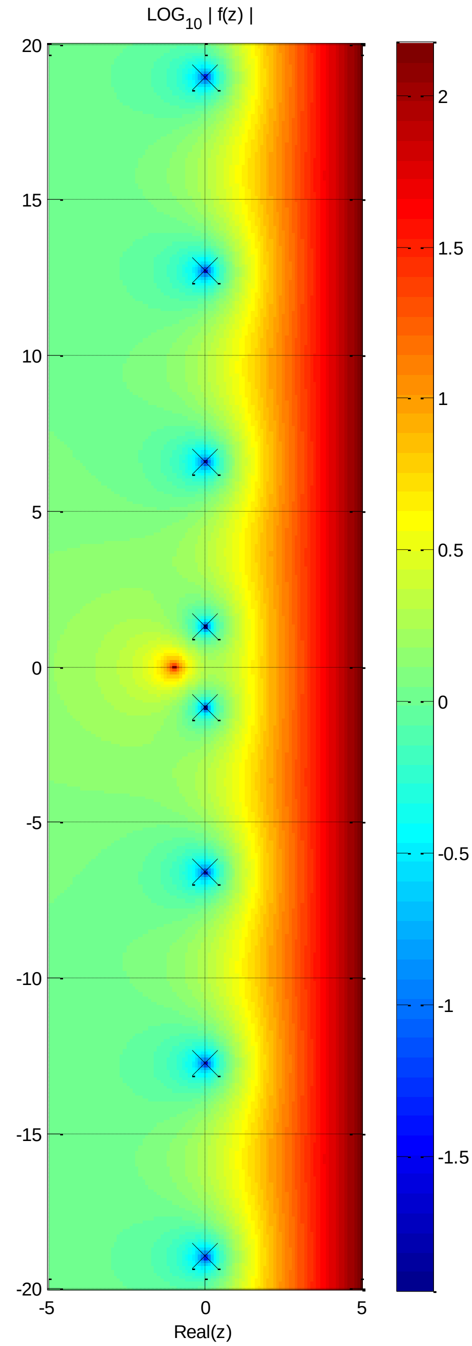
LOG$_{10}$ | f(z) |
Real(z)
20
15
10
5
0
-5
-10
-15
-20
-5
0
5
2
1.5
1
0.5
0
-0.5
-1
-1.5

Fig. 2

Fig. 1 shows contours of equal magnitude (blue; logarithmically spaced) and equal phase (red) of the function $f(z) = e^z - \frac{z-1}{z+1}$. Contours of equal magnitude encircle the zeros (roots) and poles of $f(z)$, and contours of equal phase issue from poles and zeros. There is a pole of $f(z)$ at $z = -1$. Fig. 2 shows a color map of $|f(z)|$ in logarithmic scale and the markers (x) show the roots listed in Table 2.