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RE: Puzzle Corner, Technology Review, Problem J/A 2

J/A 2. $n \geq 1$ hats belonging to $n$ people are mixed up in a cloak room. Let $a_n$ be the number of permutations of hats in which nobody gets the right hat. A corollary of the definition of $a_n$ is the recursion relation:

$$a_n = n! - \sum_{k=1}^{n} \binom{n}{k} a_{n-k}$$

(1)

since $a_n$ equals the total number of permutations less the number of permutations in which exactly $k = 1, 2, \ldots, n$ people get the right hat. We introduced the notation $a_0 = 1$ in (1) so that for one hat, one owner, $a_1 = 0$.

The probability that nobody gets the right hat is $p_n = a_n / n!$ if all $n!$ permutations are equally likely. Using the notation $p_0 = 1$, and dividing both sides of (1) by $n!$, we obtain:

$$p_n = 1 - \sum_{k=1}^{n} \frac{p_{n-k}}{k!}$$

(2)

Recursion relations (1-2) are sufficient to calculate $a_n$ and $p_n$ as shown in the table below:

<table>
<thead>
<tr>
<th>$n$ = number of hats</th>
<th>$a_n$ = number of permutations in which no hat is right</th>
<th>$p_n = a_n / n!$ = probability that no hat is right</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1/3</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>3/8</td>
</tr>
<tr>
<td>5</td>
<td>44</td>
<td>11/30</td>
</tr>
<tr>
<td>6</td>
<td>265</td>
<td>53/144</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n \rightarrow \infty$</td>
<td>...</td>
<td>$e^{-1}$</td>
</tr>
</tbody>
</table>

Let $p = \lim_{n \rightarrow \infty} p_n$. It follows from the recursion relation (2) that:

$$p = 1 - p \left( \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \right) = 1 - (e-1)p = e^{-1}$$

(3)
The solution to the recursion relation (2) is the $n$-term partial sum in the Taylor expansion of $e^{-1}$:

$$
p_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}
$$

(4)

To obtain this solution, we define $d_0 = 1$; $d_n = p_n - p_{n-1}$ for $n > 0$, and substitute the expansions of $p_n$ and $p_{n-1}$ from (2). This yields:

$$
0 = d_0 + \frac{n!}{(n-1)!} d_1 + \frac{n!}{(n-2)!} d_2 + \ldots + \frac{n!}{2!} d_{n-2} + \frac{n!}{1!} d_{n-1} + n! d_n
$$

(5)

It is readily seen that (5) is satisfied by $d_k = (-1)^k / k!$ by using the binomial expansion:

$$
0 = (1 - 1)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{(-1)^k}{k!}
$$

(6)

The probability $p_n$ is obtained by summing the differences: $p_n = \sum_{k=0}^{n} (p_k - p_{k-1}) = \sum_{k=0}^{n} d_k$ from which (4) follows.