The two sequences $x_n$ and $y_n$ satisfy the same linear second order difference equation below (*) but with different initial conditions, $x_1 = 1$, $x_2 = a$; $y_1 = 1$, $y_2 = (a - 1)$. Since the determinant
\[
\begin{vmatrix}
  x_1 & x_2 \\
  y_1 & y_2 \\
\end{vmatrix} = -1 \neq 0
\]
the two sequences are linearly independent.

The simplest way to find these sequences is to express them in terms of two linearly independent solutions, $w_n$ and $z_n$, which satisfy the same difference equation (*), with the following initial conditions: $w_1 = 1$, $w_2 = 0$; and $z_1 = 0$, $z_2 = 1$. Once these sequences have been determined, the unknown sequences may be written, using their initial conditions, as:

1. $y_n = w_n + (a - 1)z_n$
2. $x_n = w_n + az_n$

By subtracting (2) from (1) we find

\[
y_n = x_n - z_n
\]

which relates (answering Ornstein’s request) the two sequences $x_n$ and $y_n$ in terms of the sequence $z_n$ which satisfies

(*) $z_n = az_{n-1} - z_{n-2}$ with $z_1 = 0$ and $z_2 = 1$ as indicated above.

For the cases $a > 2$, [the case $a = 2$ will be considered later] we assume a solution of the form $z_n = Ct^n$, where $C$ is an arbitrary constant, and two values of $t$ ($h&k$) are determined by solving a quadratic equation. Thus, we find

$z_n = Ah^n + Bk^n$

where $2h = a + b$, $2k = a - b$ and $b = \sqrt{(a^2 - 4)}$. $A$ and $B$ are constants to be determined by applying the initial conditions. Carrying this out, we find

$z_{n+1} = (1/b)[h^n - k^n] = (1/2^n)(1/b)[(a+b)^n - (a-b)^n]$

After some algebra and replacing $b$ with $\sqrt{(a^2 - 4)}$, we can write this result in terms of the sums:

\[
even \quad z_{2(n+1)} = (1/2^{2n}) \sum_{s=0,1,2}^n [(2n+1)/(2n-2s)(2s+1)!]a^{2n-2s}(a^2 - 4)^s
\]

\[
odd \quad z_{2n+1} = (a/2^{2n-1}) \sum_{s=1,2,3}^n [(2n)/(2n-2s+1)(2s-1)!]a^{2n-2s}(a^2 - 4)^{s-1}
\]

Note the exceptional case when $a = 2$, for which $h = k = 1$, i.e., only one solution for $t$ is found. In this case the general solutions to (*) are of the form $(A + Bn)$ where $A$ and $B$ are arbitrary constants.

Applying the initial conditions we find

\[
y_n = 1 \quad \text{and} \quad x_n = n \quad \text{for} \quad n = 1, 2, 3, \ldots
\]

Thus,

\[
x_n = ny_n \quad \text{for the special case} \quad a = 2.
\]