ALLAN J. GOTTLIEB, '67

An Antepenultimate Adieu

I am sorry to say that I must relate some bad news. A few weeks ago I received a call from Susan Lewis, a senior editor at *Technology Review*, who told me that *TR* would unfortunately have to phase out "Puzzle Corner" by printing no new problems. Apparently, there is pressure for more coverage of MIT affairs and reports about MIT graduates, and at the same time, there are budget and other restrictions on the number of pages in the magazine. As Ms. Lewis accurately stated: If a subscriber wishes to find mathematical puzzles, there are alternative publications; for news about MIT alumni, there is no other source. As a result, *TR* feels compelled to terminate publication of "Puzzle Corner." Ms. Lewis added that the decision was made with a heavy heart and that those involved with the decision realized that many long-time readers of this column have formed a kind of extended family.

Although I am sympathetic with the space problems and cannot argue that there are alternative avenues for presenting alumni news, I am naturally disappointed with having "Puzzle Corner" phased out. The column started 24 years ago in *Tech Engineering News* and has appeared for 23 years in *Technology Review*. I would guess that it is one of the longest running columns in existence and I do feel sorry that I am losing contact with several long-time friends, most of whom I have never had the privilege of meeting.

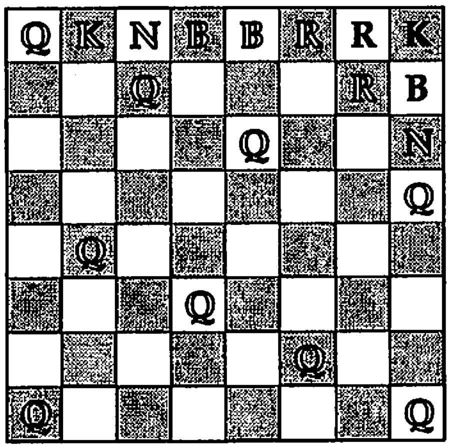
There will be two issues after this one, at which point answers will have been published for all previously posed problems. I will save my final goodbye for the last column.

Solutions

F/M 1. We begin with a newly arrived chess problem from Warner Smith. Find a legal chess position with the largest possible number of successors, i.e., for which the side to move has the largest possible number of moves.

Matthew Fountain sent us a pedestrian chess configuration:

Just like in my most recent game with Kasparov, White has 9 queens and 203 legal moves. With all those choices, no wonder I did not pick the right move and let the champ win again.



Also solved by Howard Sard, Steven Feldman, and the proposer.

F/M 2. Jerry Grossman wants you to find two irrational numbers r and s such that r^s is rational and then find two other irrational numbers r and s such that r^s is irrational.

Gerald Leibowitz begins by asserting (the well-known fact) that e is transcendental. Hence, ln2 is irrational (otherwise, there would be positive integers m and n such that $e^m = 2^n$). But $e^{\ln 2} = 2$ is rational. Also $e^{1+\ln 2} = 2e$ is irrational.

Also solved by Howard Stern, Robert Rorschach, Bob High, Ken Rosato, Gordon Rice, Harry Zaremba, Lyman Hurd, Tom Harriman, Matthew Fountain, and Walter Nissen.

F/M 3. Norman Megill has a cute version of our yearly problem. Although, when you read this problem, it will be 1990, I have kept it as 1989. In formal number theory, there are three primitive operations, + (plus), × (times), and S (successor or "1 plus"), along with a single primitive number, 0 (zero). Any positive integer can be represented by a combination of 0 and the three operations. For example, 67 can be represented by 0 preceded by 67 S's (68 symbols) or more compactly by S(SSS0 × SS(SSSSS0 × SSSS0)) (20 symbols), meaning (1 + (3 × (1 + 1 + (5 × 4))). Parentheses are excluded when counting symbols. What is the shortest representation for the number 1989?

The following solution is from Bob High (a copy of his program is available from the editor upon request):

Let the minimal length of a representation of n using the successor, addition, and multiplication operations, as defined in this problem, be known as the *complexity* of n. I wrote a little program to recursively enumerate the complexities of all $n<N$ for a given N. The complexity of 1989 is 34, and the minimal expression is 1989 = SSS0 × SSS0 × S(SSSS0 × SSSS0) × S(SSS0 × SSSS0), from 1989 = 3 × 3 × 17 × 13. In examining the resulting table, a number of interesting observations and conjectures arise. For *none* of the first 10,000 numbers n does the minimal expression involve addition! I conjecture that addition will *never* appear in a minimal expression.

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DIAGRAM: DAVID HRUBY

July

CX(N)	FIRST	LAST	CX(N)	FIRST	LAST	CX(N)	FIRST	LAST	CX(N)	FIRST	LAST
1	0	0	14	22	30	27	263	625	40	3119	12500
2	1	1	15	23	36	28	347	768	41	4283	16384
3	2	2	16	33	48	29	383	1024	42	5534	20480
4	3	3	17	38	64	30	479	1280	43	6299	25600
5	4	4	18	47	80	31	538	1600	44	7559	32000
6	5	5	19	58	100	32	718	2000	45	9358	40000
7	6	6	20	59	125	33	719	2500	46	12473	NA
8	7	7	21	83	150	34	998	3125	47	15118	NA
9	8	9	22	107	192	35	1319	4096	48	17638	NA
10	10	12	23	134	256	36	1438	5120	49	23039	NA
11	11	16	24	158	320	37	1439	6400	50	26459	NA
12	14	20	25	167	400	38	2579	8000	51	33599	NA
13	19	25	26	179	500	39	2879	10000	52	44159	NA

F/M 3 Figure

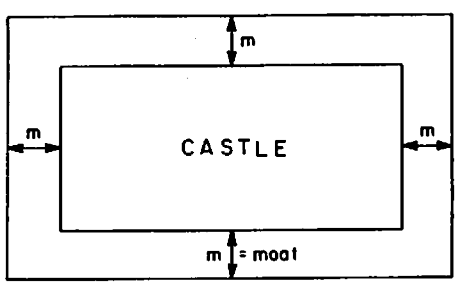
This conjecture is made plausible by the empirical observation that the complexity function, call it "cx," grows so slowly with n that it is "strongly subadditive", i.e., that $cx(a + b) < cx(a) + cx(b)$ for all a and b. This conjecture is strengthened by the fact that the complexity of n is bounded by $5\log_2(n)$, since if we write n in binary as $n = a_k a_{k-1} a_{k-2} \ldots$, we can always express it as S(SS0 [× SS0 . . .] × S(SS0 [× SS0 . . .] × S(. . .). .)); for example, 7 = 111 = S(SS0 × S(SS0)). One can come up with even stronger bounds for large n, but this is good enough. Another interesting question is to give a good *lower bound* for the complexity function.

I include (above) a table of the first and last (highest) occurrences of numbers n of a given complexity up to $n = 50{,}000$. This table was calculated under the assumption that addition can be ignored, as this tremendously speeds the computation. (The first 10,000 values were calculated initially including the "addition option.")

It's also interesting to look for numbers n the simplest expression of which starts with multiple successor operations; for example, the simplest expression for 179 is as SSSS(175). Are there numbers with arbitrarily long runs of successors in their simplest expression? Another interesting question is how much simpler than $n - 1$ n can be; for example, 1440 has a complexity of 31, while 1439 has a complexity of 37, a "jump" of 6 (the largest jump I found among all n under 50,000). Not surprisingly, the simplest expression of 1439 is as SSSS(1435). I'd be interested to know the answers to any of these questions, if the propounder, Norman Megill, or anyone else, knows them.

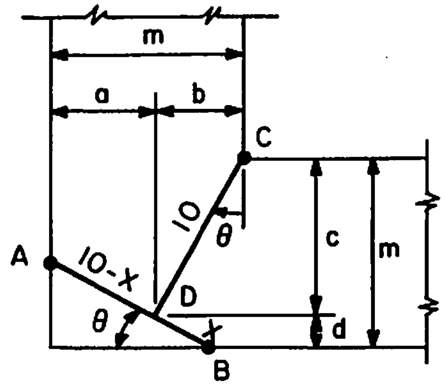
Also solved by Steven Feldman, Robert Rorschach, Ken Rosato, Gordon Rice, Harry Zaremba, Avi Ornstein, Roland Roberts, Allan Wiegner, John Woolston, Forrest Darrough, Jr., Bernie (no last name given), Matthew Fountain, Stuart Scharf, David Detlets (spelling approximate), and the proposer.

F/M 4. Gary Schmidt and Joseph Horton have a castle surrounded by a rectangular moat and have two 10-foot boards to use to cross the moat.



The boards cannot be nailed or glued together. What is the widest moat they can cross?

The following solution is from Harry Zaremba: The moat can be crossed by positioning the two boards, AB and CD, as shown in the figure at upper right.



The boards are assumed to be supported at their extreme ends and placed perpendicular to each other. From the geometry, a + b = c + d or, $(10 - x)\cos\theta + 10\sin\theta = 10\cos\theta + x\sin\theta$.

solving for x, $x = \dfrac{10\sin\theta}{\sin\theta + \cos\theta}$

The moat's width is given by,

$$m = c + d = 10\cos\theta + x\sin\theta$$

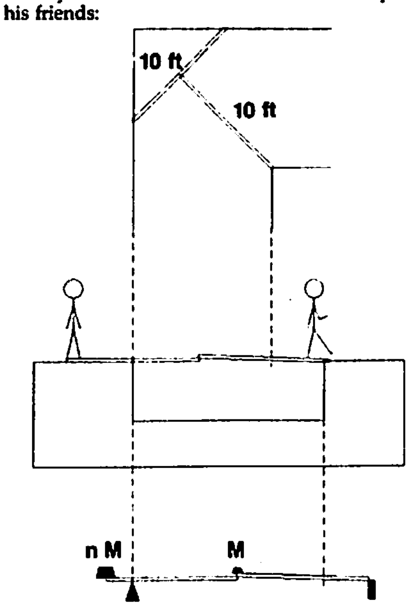
substituting x into above,

$$m = \frac{10(1 + \sin\theta\cos\theta)}{\sin\theta + \cos\theta}$$

Maximum value of m occurs when $\theta = 45°$; hence, the largest width will be, $m = 7.5\sqrt{2} = 10.6066$ ft.

The widest moat corresponds to placing board AB with its ends at equal distances from the intersection of the moat's sides, and positioning board CD to span from the midpoint of AB to the castle corner at C.

Willy Burke can do better with a little help from his friends:



Berke concludes that with enough help the maximum moat width can be just under 20 feet but once there are more than 3 or 4 helpers the anchor end of the board gets pretty crowded.

Also solved by Steven Feldman, Bob High, Ken Rosato, Gordon Rice, Lyman Hurd, Avi Ornstein, Roland Roberts, Allan Wiegner, John Woolston, Art Hall, Gwen Yueh, Allan Yueh, Sidney B. Williams, Kelly Woods, Bernie (no last name given), Eugene Sard, Mary Lindenberg, Tom Harriman, Matthew Fountain, Fredric Berger, and the proposer.

F/M 5. Our final regular problem is from that famous riverboat gambler, Gordon Rice. (A) In a gambling game, there are eight balls, numbered 1 through 8, which are shaken up in a jar, and then poured into a funnel leading to a vertical glass tube where they stack on top of each other. Wagers are made, for example, that the 2-ball will be higher in the stack than the 1-ball (even money), or that the 3-ball will be higher than the 2-ball. Another player wants to bet that both things will not happen in a single trial, and offers you 4-to-1 odds. Is this a fair bet? If not, who has the advantage? (B) Unbeknownst to the other players, a cheater succeeds in palming the standard 1-ball and substituting another which is twice as heavy. How should he then bet to take advantage of his trick?

The following solution is from Lyman Hurd:

In all these problems balls 4-8 are irrelevant and will be ignored. The sample space consists of six elements and in the fair game the probabilities of each of the six occurrences in 1/6.

First introduce some notation:

Denote by E_A the expected return on a unit bet that the 2-ball will be higher than the 1-ball, E_B the expected return on betting that the 3-ball will be higher than the 2-ball (both bets at even money), and E_C the expected value of betting at 4-to-1 odds that neither happens.

(A) Clearly the even-money bets are fair ($E_A = E_B = 0$). In the case of the third bet, only one of the six possible dispositions of the six balls pays off. The third bet has an expectation $E_C = (1/6)(4) - (5/6)(1) = -1/6$. You should not take him up on his offer (i.e., bet that at least one of the two events *will* happen. If the odds are adjusted to 5-to-1, the expectation $E'_C = 0$.

(B) The new weighting of the balls clearly has affected the distribution, but sticking to mathematics rather than physics, it is not immediately clear by how much the odds have shifted. What is clear is that the first ball can occupy three positions (relative to the other two) and that they have probabilities $a \le b \le c$, where c denotes the probability that the 1-ball is on the bottom, and a that it is on the top. Writing the bottom ball on the right, the relative weights now become:

$$P(123) = a/2 \qquad (1)$$
$$P(132) = a/2 \qquad (2)$$
$$P(213) = b/2 \qquad (3)$$
$$P(312) = b/2 \qquad (4)$$
$$P(231) = c/2 \qquad (5)$$
$$P(321) = c/2 \qquad (6)$$

The cheat clearly has not affected the second bet.

$$E_B = (a/2 + b/2 + c/2)(1) - (a/2 + b/2 + c/2)(1) = 0$$

For the first bet, the new expected value is:

$$E_A = (b/2 + c)(1) - (b/2 + a) = c - a > 0$$

which is now favorable. For the third bet the expected value is:

$$E_C = (a/2)(4) - (a/2 + b + c)(1) = (3/2)a - b - c < 0$$

The question is whether it is more favorable to bet that the 2-ball is higher than the 1-ball at even money, or to bet that the 2-ball is higher than the 1-ball or the 3-ball is higher than the 2-ball at 1-4 odds.

It hinges on the relative sizes of E_A and $-E_C$. But

$$E_A - (-E_C) = (c - a) + ((3/2)a - b - c) = a/2 - b < 0$$

Therefore, the greatest advantage is to be had by choosing the advantageous direction on bet C.

Also solved by Bob High, Harry Zaremba, John Woolston, Willy Burke, Tom Harriman, Matthew Fountain, and the proposer.

Better Late Than Never

1989 F/M 3. Forrest Darrough, Jr., suggests using a 1.5-inch-wide saw blade.

N/D 1. Richard Hess has responded.

N/D 2. Richard Hess has responded.

N/D 3. Richard Hess has responded.

N/D 4. Richard Hess and Randall Whitman have responded.

N/D 5. Richard Hess has responded.

1990 JAN 2. Jim Dorsey and Thomas Sico have responded.

JAN 4. Edgar Rose has responded.