Re: Moat Control

David Chandler writes that seeing his brother's solution in print has inspired him to submit answers to the October problems.

Bob High has kindly offered to send us some Go-related puzzles to rotate with the chess and computer offerings: Mr. High is "an officer of the American Go Association and always on the lookout for new ways to promote the game."

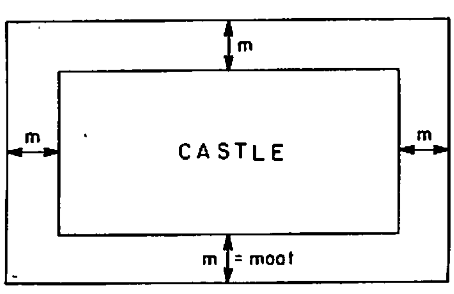
Problems

F/M 1. We begin with a newly arrived chess problem from Warner Smith. Find a legal chess position with the largest possible number of successors, i.e., for which the side to move has the largest possible number of moves.

F/M 2. Jerry Grossman wants you to find two irrational numbers *r* and *s* such that r^s is rational and then find two other irrational numbers *r* and *s* such that r^s is irrational.

F/M 3. Norman Megill has a cute version of our yearly problem. (Although when you read this problem it will be 1990, I have kept it as 1989.) In formal number theory, there are three primitive operations, + (plus), × (times), and S (successor or "1 plus"), along with a single primitive number, 0 (zero). Any positive integer can be represented by a combination of 0 and the three operations. For example, 67 can be represented by 0 preceded by 67 S's (68 symbols) or more compactly by S(SSS0 × SS(SSSSS0 × SSSS0)) (20 symbols), meaning (1 + (3 × (1 + 1 + (5 × 4))). Parentheses are excluded when counting symbols. What is the shortest representation for the number 1989?

F/M 4. Gary Schmidt and Joseph Horton have a castle surrounded by a rectangular moat and have two 10-foot boards to use to cross the moat.



The boards cannot be nailed or glued together. What is the widest moat they can cross?

F/M 5. Our final regular problem is from that famous riverboat gambler, Gordon Rice. (A) In a gambling game, there are eight balls, numbered 1 through 8, which are shaken up in a jar, and then poured into a funnel leading to a vertical glass tube where they stack on top of each other. Wagers are made, for example, that the 2-ball will be higher in the stack than the 1-ball (even money), or that the 3-ball will be higher than the 2-ball. Another player wants to bet that both things will not happen in a single trial, and offers you 4-to-1 odds. Is this a fair bet? If not, who has the advantage? (B) Unbeknownst to the other players, a cheater succeeds in palming the standard 1-ball and substituting another which is twice as heavy. How should he then bet to take advantage of his trick?

Speed Department

SD 1. Speedy Jim Landau wants you to interpret "No one under C_8H_{18} admitted!"

SD 2. L.R. Steffens needs to punctuate 123456789 so that it will describe an event that happened twice in Times Square during this century and will also happen twice in the next century.

Solutions

OCT 1. We begin with a bridge problem from Tom Harriman.

NORTH
♠ A J 7 5
♥ A Q 7 4 2
♦ A J
♣ 5 4

WEST
♠ 8 6
♥ 10 8 2
♦ 10 8 7 5 3
♣ 9 8 7

EAST
♠ Q 10 9 2
♥ K 9 6
♦ K Q 6 4 2
♣ 10

SOUTH
♠ K 4 3
♥ J 5
♦ 9
♣ A K Q J 6 3 2

How does South play to make seven clubs against best defense after the opening lead of the spade 8?

The following solution (his first submitted in 17 years of reading *Tech Review*) is from Andy Wasserman:

A quick look at declarer's assets show eleven tricks off the top, with potential losers in spades and hearts. Since we spurn losing finesses, and since throw-ins are seldom effective in grand slams (unless we can convince an opponent to revoke), we immediately consider a squeeze. Moreover, needing two tricks, we need a repeating squeeze—in this case, a repeating triple squeeze against East.

If a triple squeeze is going to repeat against the best defense, there must be two threats in the upper hand (South). Otherwise, on the squeeze card, the defender being squeezed can simply establish the threat in the upper hand. This gives declarer one trick, but not two, since declarer is left with two threats in the lower hand (North). When declarer then plays his established winner, he must discard from the lower hand before the defender, squeezing himself.

In addition, as with any squeeze, there must be an entry to the established threat. So, even with two threats in the lower hand, the best defense can prevent a triple squeeze from repeating by establishing the lower threat if there is no entry to one of the upper threats.

So, to make this hand via a repeating triple squeeze against East, all we need is one threat in the North hand accompanied by an entry in its suit and two threats in the South hand accompanied by an entry in one of the two suits. It then becomes obvious to the most casual observer that the threats must be the jack of diamonds, jack of hearts, and four of spades. The ace of diamonds provides the entry to the North hand, but the only entry to the South hand outside of trump is the king of spades. Ergo, the king of spades must be preserved at all costs.

Therefore, the key play comes at the first trick, at which point South must rise with North's ace of spades. This is followed by the ace of hearts—Vienna Coup to establish the jack as a threat—and six clubs. On one of the long clubs, North must discard the queen of hearts.

The position at this point is:

North
♠ 5
♥ 4 2
♦ A J
♣ —

West
Irrelevant

East
♠ Q 10
♥ K
♦ K Q
♣ —

South
♠ K 4
♥ J
♦ 9
♣ 2

On the last club, North can pitch a heart, but East is finished. If he pitches a spade, South cashes two spade winners, throwing a heart from North, and inflicts a red suit simple squeeze for the thirteenth trick. If East pitches his heart, South cashes his heart winner and squeezes East in spades and diamonds. Finally, if East pitches a diamond, South crosses in diamonds, and the jack of diamonds squeezes East in the majors.

Two things are noteworthy. First, the triple squeeze functions against East because West holds only two spades. Giving West the deuce of spades and removing a card from any other suit kills the

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squeeze, because West's spade eight takes care of South's threat and allows East to unguard the suit. Second, an opening diamond lead defeats the contract. A triple squeeze will still work against East, but it will not repeat because it removes a key entry from North. East will unguard spades, giving South one trick, but the resulting red suit squeeze against East fails for lack of entries.

Also solved by Francis Leahy, Amy Lowenstein, Robert Zier, Alex Halberstadt, John Stiehler, Robert Bart, Chip Palmer, David Chandler, John Chandler, Charlie Larson, Winslow Hartford, Donald and Nancy Noble, Eugene Biek, Richard Hess, Carey Rappaport, Matthew Fountain, Dan Frankel, and the proposer.

OCT 2. Gordon Rice supposes that some time in the (not too distant?) future, the art of pencil and paper arithmetic has been forgotten. Also, your computer is giving off smoke. With no way to add, subtract, multiply, or divide except an 8-digit calculator, can you evaluate the following expressions?

$$3\cdot180997^2 - 313496^2$$
$$3\cdot37467^2 - 64896^2$$

The following solution is from my children's honorary grandfather, Phelps Meaker:

$$\begin{aligned} 3\cdot180997^2 &= 3(180^2\cdot1000\cdot1000+2\cdot180\cdot1000\cdot997+997^2)\\ &= 3(32{,}400\cdot1000\cdot1000+358{,}920\cdot1000+994\cdot1000+9)\\ &= 98{,}279{,}742\cdot1000+27\\ 313496^2 &= 313^2\cdot1000\cdot1000+2\cdot313\cdot1000\cdot496+496^2\\ &= 97{,}969\cdot1000\cdot1000+310{,}496\cdot1000+246\cdot1000+16\\ &= 98{,}279{,}742\cdot1000+16\\ 3\cdot180997^2 - 313496^2 &= 98{,}279{,}742\cdot1000-98{,}279{,}742\cdot1000+27-16=11 \text{ Q.E.D.}\end{aligned}$$

$$\begin{aligned} 3\cdot37467^2 &= 3(374^2\cdot100\cdot100+2\cdot374\cdot100\cdot67+67^2)\\ &= 3(139{,}876\cdot100\cdot100+50{,}116\cdot100+44\cdot100+89)\\ &= 42{,}113{,}280\cdot100+267\\ 64896^2 &= 648^2\cdot100\cdot100+2\cdot648\cdot100\cdot96+96^2\\ &= 419{,}904\cdot100\cdot100+124{,}416\cdot100+92\cdot100+16\\ &= 42{,}114{,}908\cdot100+16\\ 3\cdot37467^2 - 64896^2 &= (42{,}114{,}908\cdot100-42{,}113{,}280\cdot100+16-267)(-1)\\ &= (1628\cdot100+16-267)(-1)\\ &= -162{,}549 \text{ Q.E.D.}\end{aligned}$$

Also solved by Robert Bart, David Chandler, Jock Young, Harry Garber, John Chandler, Charlie Larson, Thomas Harriman, Jim Martin, Winslow Hartford, Richard Hess, Carey Rappaport, Matthew Fountain, Bob High, Dan Frankel, and the proposer.

OCT 3. John Rule has a three-digit number that, when divided by the product of its digits, yields as quotient the hundredth digit. Rule wants you to find this number and show that it is unique.

The following solution is from Matthew Fountain: The number is 735. Let A, B, and C represent the three digits so that $ABC = (100A+10B+C)/A$. As $10B+C < 100$, we may write $100/A < BC < (100A+100)/A^2$. BC must also be a product of two single-digit integers. When A is taken to be 9, then $11 < BC < 13$. BC has only the possibility of being 12. Noting that A^2BC equals the three-digit number, we calculate $(9^2)(12) = 972$, yielding $B=7$ and $C=2$. As $7 \times 2 = 14$, not 12, 972 is not a solution. We next try smaller values for A, having exhausted all possibilities for $A=9$. The results are in the following table.

A=8,	12<BC<15,	(64)(14)=896,
A=7,	14<BC<17,	(49)(15)=735, (49)(16)=784
A=6,	16<BC<20,	(36)(18)=648
A=5,	20<BC<24,	(25)(21)=525
A=4,	25<BC<32,	(16)(27)=432, (16)(28)=448
A=3,	33<BC<45,	(9)(35)=315, (9)(36)=324, (9)(42)=378
A=2,	50<BC<75,	(4)(54)=216, (4)(56)=224, (4)(63)=252, (4)(64)=256, (4)(72)=288
A=1,	100<BC<200	

The only case where the product of the last two digits of a tabulated product equals BC is $A=7$, $BC=15$, and $(49)(15)=735$. We do not need to test BC's that are a multiple of 10 or contain prime factors larger than 9.

Also solved by Edward Dawson, Robert Bart, David Chandler, Harry Garber, Mary Lindenberg, Leonard Nissim, Thomas Harriman, Winslow Hartford, Donald Savage, Avi Ornstein, Richard Hess, Carey Rappaport, Gordon Rice, Steven Feldman, and Bob High.

OCT 4. David Evans notes that on an 8×8 checkerboard, if two squares of the same color are removed, it is impossible to cover the remaining 62 squares with 31 1×2 tiles (since each tile covers one white and one black square). Is the converse true, i.e., if you remove 2 squares of opposite colors, can the remaining 62 squares always be covered by 31 1×2 tiles?

The following solution is from Al Zobrist:
Let "coverable" be the property that any 2 squares of the opposite color can be removed and the board can be covered with 1×2 tiles. A 2×2 checkerboard is obviously coverable. Suppose that an $n \times n$ checkerboard (n even) is coverable. Augment this to the next larger even-sided checkerboard by surrounding it with a single layer of squares. Three cases show that the larger checkerboard is coverable.
1. The removed squares are both interior. Cover the interior (it is $n \times n$), and cover the outside (it has $4n + 4$ squares arranged linearly).
2. The removed squares are both exterior. Cover the interior (each row is even and linear). The exterior removed squares are opposite in color, hence are separated by even numbers of squares both ways.
3. One of each. Suppose the interior removed is white. Remove another interior black and an adjacent exterior white. Cover the interior as in case 1 and the exterior as in case 2, then unremove the additional removed squares and cover them with one tile.

By induction, any $n \times n$ checkerboard is coverable if n is even and positive.

Also solved by Robert Bart, Jim Roskind, David Chandler, Richard Hess, Thomas Harriman, Winslow Hartford, Carey Rappaport, Matthew Fountain, Gordon Rice, Ken Rosato, Bob High, and the proposer.

OCT 5. Chuck Coltharp poses the following partitioning question. Let S be a finite set of size 4n and let P be a collection of partitions of S, each of which partitions S into two disjoint sets of size 2n. Let the i^{th} partition be the two sets A_i and B_i. We require that, for $i \neq j$, $A_i \cap B_j$ is of size n. The question is how large can P be, that is for each n what is the largest number of partitions that can be found satisfying the above properties?

I will print two solutions to this difficult problem since they have rather different characters. John Chandler submitted a direct calculation for sizes that are powers of two and Bob High related the problem to Hadamard matrices. Chandler writes:
P can be of size 4n-1 when n is a power of two. We can show this by induction. First, it is simple to see that there are exactly 3 partitions of 4 into 2 + 2 and that all three belong. Then, suppose we take a set S of size 4n and add S^* to it. $S + S^*$ has size 8n. We have 4n-1 suitable partitions of S and 4n-1 matching partitions of S^*. We can construct two full partitions from the two half-partitions by combining the two halves of each with the corresponding or opposite halves of the other, i.e., take (A_i,B_i) to be a partition of the first half and (A_i^*, B_i^*) to be a partition of the second. The two full partitions are $(A_i + A_i^*, B_i + B_i^*)$ and $(A_i + B_i^*, B_i + A_i^*)$. The intersections between pairs of these are all of size 2n, e.g., $(A_i + B_i^*) \cap (B_i + B_i^*) = B_i^*$. Similarly, we see that $(A_i + B_i^*) \cap (B_j + B_j^*) = (A_i \cap B_j) + (B_i^* \cap B_j^*)$, which is of size 2n. That gives us $2^*(4n\text{-}1) = 8n\text{-}2$ partitions of this kind, and there is one more, namely (S, S^*), and it can be seen that the intersection of S with any of the aforementioned sets has size 2n. That brings the total to 8n-1 and completes the proof by induction. For sets where n is not a power of two, the answer is a little more complicated: it is 2m-2, where m is the largest power of 2

that divides n. To see how this is so, we look again at adding two sets together, but this time sets of different sizes. If one size is a multiple of the other, the larger set will have a larger collection of partitions, but we won't be able to match them all with corresponding partitions of the smaller set. In short, we can only get twice as many partitions of the combined set as there are of the smaller set.

High's solution follows:

This turns out to be quite interesting. If P is a collection of k partitions of a set S with $4n$ members meeting the problem conditions, I can prove that the *maximum possible* value for k is $4n - 1$ (see below). However, to show that this maximum is *attained* for every n is in fact, if I am not mistaken, equivalent to an open problem in combinatorial theory!

First, let's establish the upper bound $k \leq 4n - 1$. This part of the problem has a pretty geometric interpretation: Let each partition be represented by a vector of 1's and −1's, with $2n$ 1's and $2n$ −1's in each vector. By hypothesis, any two vectors will have 1's in common in exactly n places, and −1's in common in exactly n places. Thus, for any two such vectors (a_i) and (b_i), we will have

$$\sum_{i=1}^{4n} a_i b_i = 0$$

That is, the vectors are *mutually orthogonal*.

Each partition, represented as a vector of 1's and −1's, can be considered as a point in $4n$-space. By flipping signs, if necessary, we can arrange that for a chosen $s \in S$, each vector will have a "1" in the place represented by s. We can then drop that place from all the vectors, resulting in a set of k vectors in $(4n - 1)$-space. Now, any two such vectors are *equidistant* by the conditions of the problem—they differ in exactly $2n$ places, so they are all exactly $2\sqrt{2n}$ units apart. And there is *another* point in $(4n - 1)$-space which is equidistant from each of them—the point $(1, \ldots, 1)$. These $k + 1$ points therefore form the vertices of a k-simplex. This shows immediately that there can't be more than $(4n - 1)$ such partitions, since one can't embed a $4n$-simplex in $(4n - 1)$-space!

This says that if we *can* find a collection of $k = 4n - 1$ partitions, they will give rise to a $(4n - 1)$-simplex embedded in the $(4n - 1)$-dimensional cube. Conversely, if we can find a simplex embedded in the $(4n - 1)$-dimensional cube, it will give us a solution to our problem. So the problem boils down to the geometric question: for which dimensions can one embed a simplex in the cube?

As far as *realizing* the maximum value of $4n - 1$, this turns out to be equivalent to the problem of the *existence of Hadamard matrices of every order* $4n$. A **Hadamard matrix** is a square m-by-m matrix A with the property that $a_{ij} = \pm 1$ for all $i\,j$ and $AA' = mI$. One can show that if A is a Hadamard matrix, then $m = 4n$. (See Ian Anderson, *A First Course in Combinatorial Mathematics, 2nd Edition*, Oxford, 1988, for example.) Now, if a Hadamard matrix exists for a given value of $m = 4n$, it proves that $k \geq 4n - 1$ in our problem (since the rows or columns of the matrix other than $(1, \ldots, 1)$, being orthogonal, define a collection of partitions meeting the problem conditions). But it is *not known* (at least, as of the publication of Anderson's book in 1988) whether Hadamard matrices exist for every possible m; according to Anderson, 428 is the first unknown case. So it's hard for me to believe that the proposer of this problem knows the answer for $4n = 428$!

It's easy to come up with Hadamard matrices (and hence solutions to our problem) for small values of n. For n = 1, for example, one has:

```
+1 +1 +1 +1
+1 +1 -1 -1
+1 -1 +1 -1
+1 -1 -1 +1
```

For n = 2,

```
+1 +1 +1 +1 +1 +1 +1 +1
+1 +1 +1 +1 -1 -1 -1 -1
+1 +1 -1 -1 +1 +1 -1 -1
+1 +1 -1 -1 -1 -1 +1 +1
+1 -1 +1 -1 +1 -1 +1 -1
+1 -1 +1 -1 -1 +1 -1 +1
+1 -1 -1 +1 +1 -1 -1 +1
+1 -1 -1 +1 -1 +1 +1 -1
```

And for n = 3, (replacing ±1 with 1 and 0 for legibility)

```
1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 0 0 0 0 0 0
1 1 1 0 0 0 1 1 1 0 0 0
1 1 0 1 0 0 1 0 0 1 1 0
1 1 0 0 1 0 0 1 0 1 0 1
1 1 0 0 0 1 0 0 1 0 1 1
1 0 1 1 0 0 0 1 0 0 1 1
1 0 1 0 1 0 0 0 1 1 1 0
1 0 1 0 0 1 1 0 0 1 0 1
1 0 0 1 1 0 1 0 1 0 0 1
1 0 0 1 0 1 0 1 1 1 0 0
1 0 0 0 1 1 1 1 0 0 1 0
```

Given one Hadamard matrix, it's easy to "bootstrap" your way to larger ones. Given any two Hadamard matrices A and B, of orders m and n respectively, just substitute A for 1 and $-A$ for −1 in B, and a new Hadamard matrix of order mn will emerge. Thus, it's easy to answer the problem completely for $n = 2^k$, for example, and for certain other values. (I had constructed these examples, and found a solution for $n = 12$, before going to Anderson.)

In speculating about the existence of a Hadamard matrix of every order $4n$, it is natural to think about the symmetries of the corresponding embedded simplex. Starting from the vertex $(1, \ldots, 1)$, take *any* other vertex v of the $4n - 1$-cube differing from this in exactly $2n$ places. If an embedded simplex exists, there should be a symmetry of the $4n - 1$-cube fixing $(1, \ldots, 1)$ and carrying the vertex v cyclically into each of the $4n - 1$ vertices of that simplex (other than $(1, \ldots, 1)$). This will be a rigid motion σ of the cube (and simplex) with the property that the distance from v to $\sigma(v)$ will always be exactly $2\sqrt{2m}$. Since such symmetries amount to permutations of the coordinate axes, this is equivalent to finding a permutation of $4n - 1$ elements with the property that for a given subset K consisting of $2n - 1$ elements, we have $K \cap \sigma^k(K)$ of size exactly $n - 1$ for all k. For example, the permutation (1425673) does the job for $n = 2$.

Also solved by David Chandler, Carey Rappaport, Matthew Fountain, and the proposer.

Better Late Than Never

1987 N/D 5. Thomas Harriman sent us a rigorous solution to this (so-called Mnage) problem. Copies are available from the editor.

1989 F/M 5. Albert Mullin notes that the largest Mersenne prime found to date is much closer to $2^{216,091} - 1$ than to $2^{19,937} - 1$ and that there are asymptotically $n/\log n$ primes less than n.

APR 4. Thomas Harriman sent us a more analytic solution; Albert Lazzarini has responded.

M/J 1. Robert Bart points out White's second move in the solution to part A is illegal and provides us with the following correct version.

	Black	White
1.	N-e4ch	K-d1
2.	N-c3ch	PxN
3.	B-e1	P-c4
4.	P-h1(R)	P-c5
5.	R-h2	P-c6
6.	RxP	P-c7
7.	R-a7	P-c8(Q) mate

JUL 2. John Prussing had responded in a timely fashion but I am sorry to say that I misplaced his letter.

JUL 3. Dimitri Daskalopoulos, L.J. Lipton, and James Conant have responded.

JUL 5. Thomas Harriman has responded.

Proposers' Solutions to Speed Problems

SD 1. An octane rating

SD 2. 1:23:45, 6/7/89. The time on June 7, 1989 am and pm, and on June 7, 2089.