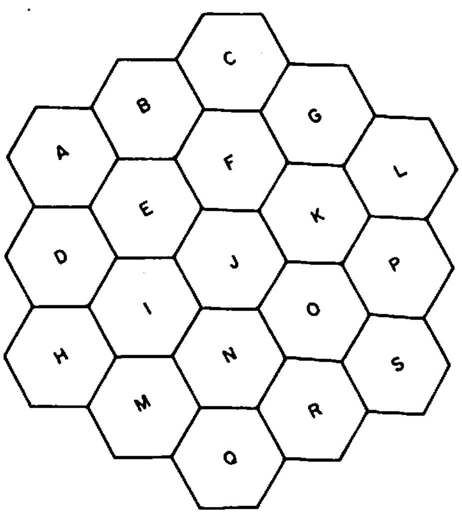
ALLAN J. GOTTLIEB, '67

Hexagony

Pretty exciting times around here. It is Friday and I am leaving on Tuesday for a conference in Israel with a post-conference tour of Egypt. However, I have only "secondhand" hotel confirmations for Israel (i.e., from the arrangement group, not from the hotels themselves), do not even know the name of the hotel in Egypt, and have not received any confirmation about the tour. This is *not* my normal style of travel, but people here tell me not to worry so I am trying not to.

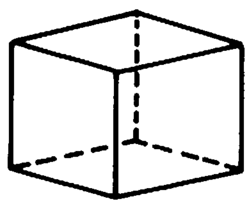
Problems

A/S 1. My plea for computer-related problems has inspired Warren Himmelberger. I still need more such problems. Mr. Himmelberger wants us to devise a computer program to use the numbers 1 through 19 once each to label the hexagons below so that each diagonal sums to the same value. Note that there are six diagonals with 3 hexagons each, six with 4, and three with 5.



A/S 2. A magic-square problem from Ronald Martin. He wants you to arrange the numbers from 1 to 121 in an 11 × 11 grid so that all rows, columns, and diagonals sum to 671. Note that to form most diagonals you must imagine a copy of the square placed next to the original.

A/S 3. Our third problem comes from a 1986 issue of *IEEE Potentials*. According to Dan Benson of the University of Washington, a plane can be found that will divide the cube into two identical halves whose common face is a regular hexagon. He says there are four such planes. How many can you find?



A/S 4. Richard Hess has taken craps to an extreme and writes: In Extreme Craps, four dice are thrown and the middle two dice are ignored. Otherwise, it is played the same as ordinary craps. What is the shooter's probability of winning Extreme Craps?

A/S 5. Frank Rubin tells us about Milo Mindbender, a student at Drudgery High. After every test, Milo figures out his cumulative average, which he always rounds to the nearest whole percent. Today he had two tests. First he got 75 in French, which dropped his average by 1 point. Then he got 83 in History, which lowered his average another two points. What is his average now?

Speed Department

SD 1. Speedy Jim Landau wants to know what happens when an irresistible force encounters an immovable object.

SD 2. Mark Astolfi has a five-card poker hand with exactly one wild card. Which two standard poker hands are impossible?

Solutions

APR 1. Our first problem, from Nob. Yoshigahara, involves multiplying a "time expression," i.e., one involving hours, minutes, and seconds by a scalar to obtain another time expression. Another requirement is that all ten digits are to be used once each.
ab:cd × e = f:gh:ij.

Steve Feldman says that he is sure he has seen (and solved) this problem sometime within the past few years but adds that he is not complaining. With a little help from a BASIC program Mr. Feldman found the correct answer

SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, THE COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y. 10012.

50:42 × 9 = 7:36:18.

Also solved by Robert Bart, John Chandler, Gordon Rice, Harry Zaremba, Richard Hess, Matthew Fountain, and the proposer.

APR 2. William Pulver knows twelve golfers who play weekly in 3 foursomes, 2 players as a team in each foursome competing against the other pair in that foursome. The problem is to arrange a schedule so that each golfer plays with each of the other eleven the same number of times and against each of the eleven the same number of times.

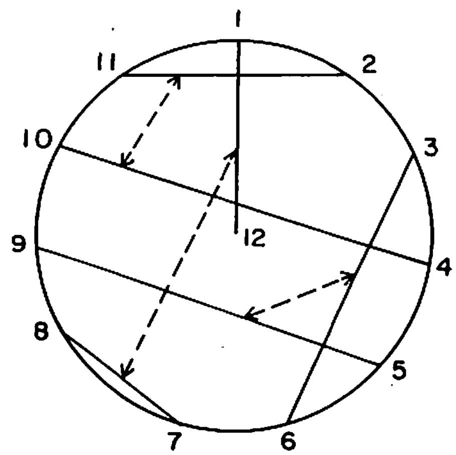
The following solution is from Matthew Fountain: The schedule below is satisfactory.

1,12- 7,8	2,11- 4,10	5,9 - 3,6
2,12- 8,9	3,1 - 5,11	6,10- 4,7
3,12- 9,10	4,2 - 6,1	7,11- 5,8
4,12-10,11	5,3 - 7,2	8,1 - 6,9
5,12-11,1	6,4 - 8,3	9,2 - 7,10
6,12- 1,2	7,5 - 9,4	10,3 - 8,11
7,12- 2,3	8,6 -10,5	11,4 - 9,1
8,12- 3,4	9,7 -11,6	1,5 -10,2
9,12- 4,5	10,8 - 1,7	2,6 -11,3
10,12- 5,6	11,9 - 2,8	3,7 - 1,4
11,12- 6,8	1,10- 3,9	4,8 - 2,5

Each line of the schedule represents the pairings for one weekend.

The schedule is compiled by use of the accompanying diagram. Each line on the face of the circle connects two players who are partners the first weekend. Pairs of lines determine the foursomes. The pairings for the second weekend are found by rotating each number on the circumference to the next position on the circumference. Pairings for the following weekend are found by repeating this process. As none of the chords on the circle are the same length, each rotation results in pairings that do not duplicate any previous pairing.

The dotted lines indicate the two partner pairs that form a foursome. If all mutual opponents on the circumference were to be joined by lines, the ten new chords formed would be of five different lengths, two chords of each length. The first weekend the players joined by these chords would be 1–7, 1–8, 2–4, 2–10, 11–4, 11–10, 5–3, 5–6, 9–3, and 9–6. Thus as the players rotate through all the positions on the circumference they each play each other twice as opponents.



Also solved by Gordon Rice, Robert Bart, Robert Roth, and Winslow Hartford.

APR 3. Matt Stenzel wants you to show that for $p = 2, 3, 4, \ldots$

$$\frac{1}{\sum_{n=1}^{\infty} \frac{n}{p^{n+1}}}$$

is a perfect square.

I am printing two rather different solutions. Phylis Savari writes:

$$\frac{1}{\sum_{n=1}^{\infty} \frac{n}{p^{n+1}}} = \frac{1}{\frac{1}{p^2} + \frac{2}{p^3} + \frac{3}{p^4} + \frac{4}{p^5} + \ldots}$$

$$= \frac{1}{\left(\frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \ldots\right) + \left(\frac{1}{p^3} + \frac{1}{p^4} + \frac{1}{p^5} + \ldots\right) + \left(\frac{1}{p^4} + \frac{1}{p^5} + \frac{1}{p^6} + \ldots\right) + \left(\frac{1}{p^5} + \frac{1}{p^6} + \frac{1}{p^7} + \ldots\right) + \ldots}$$

and since an infinite geometric series with ratio r ($|r| < 1$) and first term a_1 is equal to

$$\frac{a_1}{1 - r},$$

we can rewrite the above expression as

$$\frac{1}{\frac{1}{p(p-1)} + \frac{1}{p^2(p-1)} + \frac{1}{p^3(p-1)} + \frac{1}{p^4(p-1)} + \ldots}$$

we can apply

$$S = \frac{a_1}{1 - r}$$

to this expression now and find that the sum of the terms in the denominator =

$$\frac{\frac{1}{p(p-1)}}{1 - \frac{1}{p}} = \frac{1}{(p-1)^2}.$$

(This is why p was constrained to 2, 3, 4, ...

If $p = 1$, $\frac{1}{(p-1)^2}$ is undefined.) Hence, the above expression can be replaced with

$$\frac{1}{\frac{1}{(p-1)^2}} = (p - 1)^2$$

which indeed is a perfect square. So

$$\frac{1}{\sum_{n=1}^{\infty} \frac{n}{p^{n+1}}} = (p-1)^2 \text{ for } p = 2, 3, 4, \ldots$$

Taking a different tack Leonard Nissim responded with:

This problem can be solved by considering the Taylor Series at the origin for the function

$$f(x) = \frac{1}{(1-x)^2} = (1 - x)^{-2}.$$

Via mathematical induction, one can prove that

$$f^{(n)}(x) = (n + 1)!(1 - x)^{-(n+2)}. \quad (1)$$

Indeed, the function as its own zero-th derivative begins the induction: $f^{(0)}(x) = (0 + 1)!(1 - x)^{-(0+2)}$. The induction hypothesis, $n = k$, is $f^{(k)}(x) = (k + 1)!(1 - x)^{-(k+2)}$. For $n = k + 1$, note that

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx} f^{(k)}(x) \\ &= \frac{d}{dx}\left((k + 1)!(1 - x)^{-(k+2)}\right) \\ &= (k + 1)!(-(k + 2))(1 - x)^{-(k+3)}(-1) \\ &= (k + 2)!(1 - x)^{-(k+3)} \end{aligned}$$

Having proved (1), one can now evaluate the Taylor Series coefficients:

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{(n+1)!}{n!} = n + 1.$$

Having determined the coefficients, the Taylor series is:

$$\sum_{n=0}^{\infty} (n + 1)x^n = 1 + 2x + 3x^2 + \ldots$$

The Ratio Test proves that this converges for all x with $|x| < 1$. One can re-index, letting $j = n + 1$. This produces:

$$\frac{1}{(1-x)^2} = \sum_{j=1}^{\infty} jx^{j-1} \text{ for } |x| < 1 \quad (2)$$

Now, for any real $p > 1$, so in particular for $p = 2, 3, 4, \ldots$, one has

$0 < \frac{1}{p} < 1,$

so that (2) may be applied with

$x = \frac{1}{p}$:

$$\sum_{n=1}^{\infty} \frac{n}{p_{n+1}} = \frac{1}{p^2} \sum_{n=1}^{\infty} n \left(\frac{1}{p}\right)^{n-1}$$

$$= \frac{1}{p^2} \frac{1}{\left(1 - \frac{1}{p}\right)^2}$$

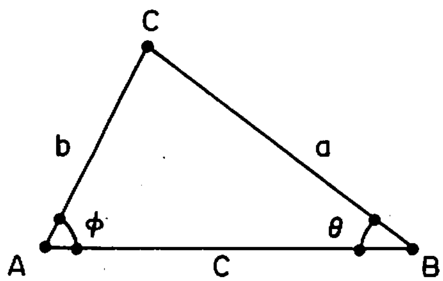
$$= \frac{1}{(p-1)^2}$$

Taking the reciprocals, one concludes:

$$\frac{1}{\sum_{n=1}^{\infty} \frac{n}{p^{n+1}}} = (p-1)^2.$$

Also solved by Robert Bart, John Chandler, N.F. Tsang, Roger Whitman, Gordon Rice, John Fogarty, Eli Passow, Stephen Persek, Henry Lieberman, Harry Zaremba, Jon Murnick, Alan Taylor, Alan Prince, Richard Hess, Matthew Fountain, and the proposer.

APR 4. Gordon Rice would like you to find non-equilateral triangles containing a 60-degree angle, all of whose sides are integers. How about 30 degrees? The following solution is from Harry Zaremba:



In the triangle shown,

$\sin\theta = \frac{b}{a} \sin\phi,$

and, $c = b\cos\phi + a\cos\theta$. Employing the identity $\cos\theta = (1 - \sin^2\theta)^{1/2}$ and substituting for $\sin\theta$, the length of side c becomes,

$c = b\cos\phi + (a^2 - b^2\sin^2\phi)^{1/2}$

For $\phi = 30°$,

$c = \frac{1}{2}(\sqrt{3}b + \sqrt{4a^2 - b^2})$

and for $\phi = 60°$,

$c = \frac{1}{2}(b + \sqrt{4a^2 - 3b^2})$

It is apparent when ϕ equals 30° that c will always be an irrational number since the term $\sqrt{3}b$ is irrational for any integer value of b. Hence, triangles with integral values for each side do not exist when one of the interior vertex angles is 30°.

When ϕ equals 60°, the number of triangles with each of their sides equal to an integer is limitless. Several solutions for integer values of c, a, and b are listed below.

a	b	c
7	3	8
19	5	21
13	7	15
31	11	35

Also solved by George Parks, Winslow Hartford, Steven Feldman, Robert Slater, Robert Bart, John Chandler, N. F. Tsang, Leonard Nissim, Stephen Persek, Richard Hess, Matthew Fountain, and the proposer.

Better Late Than Never

1987 A/S 5. Nob. Yoshigahara found a smaller solution.

1988 N/D 2. Gordon Rice believes the truth-teller could say "I know" and that the liar could say "I don't know."

N/D 3. Winslow Hartford believes "the old-fashioned day-coach paper drinking cup" when filled with water forms the shape needed. He adds, however, that "nobody knows what a day-coach is any more."

1989 JAN 2. George Parks has responded.

Proposers' Solutions To Speed Problems

SD 1. An inconceivable concussion.

SD 2. No pair and two pair.