ALLAN J. GOTTLIEB, '67

Royal Wedding

Since it has been a year since I reviewed the criteria used to select solutions for publication, let me do so now.

As responses to problems arrive, they are simply put together in neat piles, with no regard to their date of arrival or postmark. When it is time for me to write the column in which solutions are to appear, I first weed out erroneous and illegible responses. For difficult problems, this may be enough; the most publishable solution becomes obvious. Usually, however, many responses still remain. I next try to select a solution that supplies an appropriate amount of detail and that includes a minimal number of characters that are hard to set in type. A particularly elegant solution is, of course, preferred. I favor contributions from correspondents whose solutions have not previously appeared, as well as solutions that are neatly written or typed, since these produce fewer typesetting errors.

Problems

JUL 1. Robert Bart offers a problem he attributes to Robert Darves in which South is to make seven spades with the assistance of all four players.

North
♠ 8 5
♥ Q 9 7 6 5 4 2
♦ 5
♣ 4 3 2

West
♠ K 10 7 4
♥ K J 6
♦ J 10 3
♣ A Q 10

East
♠ Q 9 6 3
♥ A 10
♦ A K Q 4
♣ K J 9

South
♠ A J 2
♥ 3
♦ 9 8 7 6 2
♣ 8 7 6 5

JUL 2. As noted below, problem F/M 4 was misprinted (sorry). Here is the corrected version. *A* offers to run three laps while *B* does two but gets only 150 yards into his third lap when *B* wins. He then offers to run four laps to *B*'s three, and quickens his pace in the ratio of 4:3. *B* also quickens his pace, in the ratio of 9:8, but in the second lap falls off to his original pace, and in the third goes only 9 yards for the 10 he went in the first race. *A* wins the race by 180 yards. How long is each lap?

JUL 3. A Cryptoquiz from David Wagner:

PNUIU HC QGSL KIQFUO EUPNQO QY JCCHCPHGA PNU JOFJGRUEUGP QY KZIU CRHUGRU - PNJP QY KHRBHGA EUG QY AUGHZC, DJRBHGA PNUE NUJFHSL, JGO SUJFHGA PNUE PQ OHIURP PNUECUSFUC.

MJEUC DILJGP RQGJGP

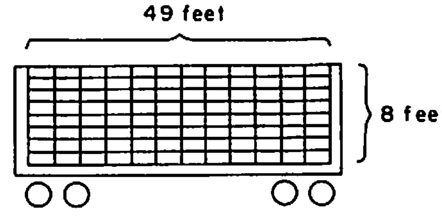
JUL 4. Matthew Fountain has been looking at "constellations," a subject previously studied by Euler. Fountain wants us to find eight distinct positive integers such that

$$A + B + C + D = E + F + G + H$$
$$A^2 + B^2 + C^2 + D^2 = E^2 + F^2 + G^2 + H^2$$
and
$$A^3 + B^3 + C^3 + D^3 = E^3 + F^3 + G^3 + H^3$$

JUL 5. Walter W. Hill and Walter L. Hill (no kidding) want to know the probability that at least one marriage occurs in a normal deck of 52 cards. A marriage is said to occur if a King and Queen are consecutive, e.g., the King of clubs is the 13th card and the Queen of hearts is the 14th.

Speed Department

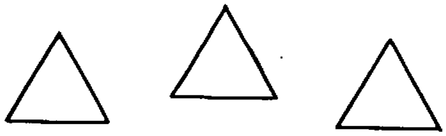
SD 1.



Jim Landau has a "bulkhead flatcar" that he has loaded with finished 2 × 4's as shown. If the lumber-carrying region of the car is 49 feet long and 8 feet high, how many 2 × 4's can the car carry?

SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, THE COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y. 10012.

SD 2.



The figure shows three triangles made from matchsticks. Nob. Yoshigahara wants you to move two matches so that no triangles remain.

Solutions

F/M 1. We begin with a bridge problem from Doug Van Patter:

North
♠ A K 5
♥ Q 4 3
♦ 7 5 4 2
♣ 8 6 5

South
♠ 7
♥ A K 10 8 6 2
♦ A K 3
♣ A 7 4

Over-optimistic bidding has led to an unlikely six heart contract. West leads the club king, which you take with the ace. When you cash the ace and king of trumps, West shows out on the second round, leaving East with the Jack. How do you proceed? If you were in a four heart contract (duplicate bridge) what line of play would you adopt?

The following solution is from Matthew Fountain: I would play my A, K of diamonds; dummy's A, K of spades while discarding my 3 of diamonds; trump a diamond lead from dummy; enter dummy with trump queen; lead a diamond and discard a club; cash my last two trumps; and concede the last trick. I figure to lose as many as three tricks if East trumps my diamond A or dummy's spade K, four tricks if East trumps my diamond K or dummy's spade A, or two tricks if defender's diamonds do not break three-three. With nine spades out I feel secure that East will not trump the first two spade leads. My probability of being set three tricks is the same as the probability that East holds one or two diamonds at the start of play, as I would change my play if he didn't follow suit at the first play of diamonds. The probability of East holding one or two diamonds is 117/323. East's probability of holding three diamonds, which allows me to make the slam, is 120/323.

At duplicate I would play the same. I understand that to win you must take chances. If you play conservative you'll always end up in the middle of the pack. It is better to adopt a style of play that makes your results swing more wildly up and down. You'll win more prizes or points toward titles that way. But this is just my conception as I have never played duplicate.

Also solved by Joe Feil, Thomas Harriman, Winslow Hartford, Jerry Grossman, and the proposer.

F/M 2. Nob. Yoshigahara sent us a problem from Y. Kotani. For each positive interger n, consider writing the integers from 1 to n inclusive and let $f(n)$ be the number of times the digit 1 was used. For example $f(3) = 1$, $f(10) = 2$, and $f(12) = 5$. Clearly $f(1) = 1$; what is the smallest $n > 1$ with $f(n) = n$?

The following clear analysis is from Bill Cane.

By inspection, $f(1) = 1$ and $f(9) = 1$. Consider the numbers from 10 to 99. They fall in two groups: those that start with 1 (10-19), and those that don't (20-99). The ten numbers that start with 1 contribute 10 ones to our function plus the same number of ones as that found from 1-9, which is $f(9)$, or 1. Hence, $f(19) = f(9) + 10 + f(9) = 12$. Those numbers that don't start with one are actually eight groups (20's, 30's, . . .), each of which has the same number of ones as the group 1-9 ($f(9)$, or 1). Together, these eight groups contribute 8 ones to our function, so $f(99) = f(19) + 8f(9) = 20$. Similar reasoning shows that the following progression will obtain:

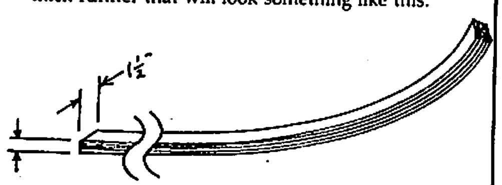
$f(199) = f(99) + 100 + f(99) = 140;$
$f(999) = f(199) + 8f(99) = 300;$
$f(1999) = f(999) + 1000 + f(999) = 1600;$
$f(9999) = f(1999) + 8f(999) = 4000;$
$f(19999) = f(9999) + 10000 + f(9999) = 18000;$
$f(99999) = f(19999) + 8f(9999) = 50000;$
and, finally,
$f(199999) = f(99999) + 100000 + f(99999) = 200000!$

Now consider 199,981. The numbers from 199,982 – 199,999 all contribute a single one to our function, except for 199,991, which contributes 2. Since there are 18 of these numbers, their total contribution to our function is 19, so we can say $f(199{,}981) = f(199{,}999) - 19 = 200{,}000 - 19 = 199{,}981$. Note that 199,981 itself contributes 2 to our function, so $f(199{,}980) = 199{,}979$. Further, all numbers from 100,000 – 199,979 each contribute *at least* 1 one to our total. Therefore, as we proceed back from 199,981, the function must decrease by at least 1 per number. We know $f(99999) = 50000$ and $f(19999) = 18000$, so there can be no number for which $f(n) = n$ that is any smaller than 199,981.

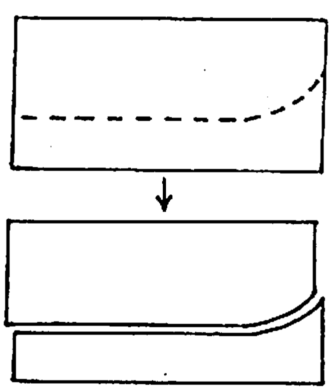
(It can be shown that $f(10^n - 1) = n10^{n-1}$ and that $f(2 \times 10^n) = 10^n + 2f(10^n)$, but I think that simply showing the table used above contributes more to the reader's understanding of the problem.)

Also solved by Gordon Rice, Harry Zaremba, Randall Whitman, Thomas Harriman, Winslow Hartford, Robert High, Matthew Fountain, Steven Feldman, Avi Ornstein, Peter Groot, Alan Taylor, and an anonymous contributer.

F/M 3. Clifford Cantor, from Anchorage Alaska, asks our first ever dogsled-theoretic problem. My brother Jim is laminating together four thin layers of hickory to make a new runner for his dogsled. Each piece is 0.25" thick, 1.5" wide, and about 8' long. The four pieces will be laminated to form a 1" thick runner that will look something like this.



He wants to make a 2-piece mold for gluing the strips together. He plans to cut an 8' sheet of plywood 1.5" thick, so the cut edges will be the surfaces of the mold.



He called me to ask whether he can produce both pieces with a single saw cut. I told him he could not. Was I right? If not, what curve would have the requisite properties?

Norman Spencer solved this one and then wrote: Along with regular Bridge problems, are we now going to see regular challenges to a basement woodworker? My wife has trouble enough understanding why I make so much sawdust in the basement, but watching me work on this problem convinced her of my derangement. Thanks for the fun.

Mr. Spencer's solution follows:

We are asked to design a mold for a laminated dog-sled runner with a ski-tip front, with the limitation of a single cut across a sheet of plywood to form the two halves of the mold. The runner is 1" thick.

In Bruce Hoadley's *Understanding Wood*, we are told that the deformation resulting from successful steam-bending is mostly compression of the concave side, with negligible lengthening of the outer curve. It follows that the geometric problem posed is that the inner-mold requires a radius of curvature of one inch less than the outer curve at any radial position. I assume that the runner is straight for most of its length and the front ski-tip curve covers no more than 90° of arc.

For a mathematically simpler solution, make the infinitely thin band-saw cut as drawn in Fig. 1 to form a ski-tip front with constant radius of curvature. The angles of the arcs at either end of the cut are equal, and are defined by the design. The mold functions by flipping the upper half onto its reverse side shifting it to the left by up to one-inch, and forming the ski-tip at the right side. The left hand curved portions of the mold are non-functional, but will the molds fit? That is, will the tighter curve on the left end of the lower mold overlap the upper mold once the upper half has been flipped and positioned one inch above the lower half?

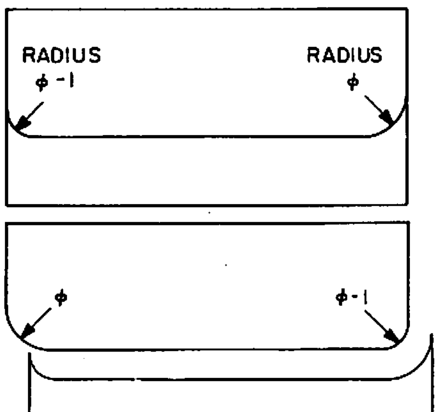


Fig. 1

Let the angular arc of the mold curve at each end be α. See Fig. 2.

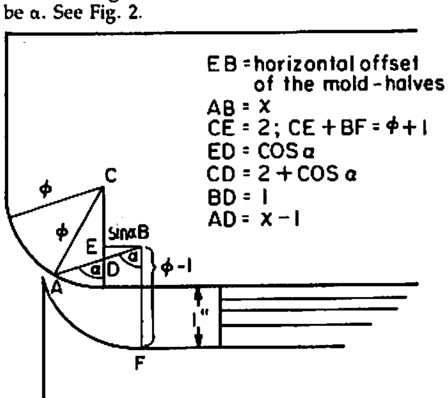


Fig. 2

The molds will fit only if $0 \leq x \leq \phi - 1$, or restated, $x + 1 \leq \phi$. Necessarily, $\phi > 1$

From the Law of Cosines for triangle ACD:

$$\phi^2 = (2 + \cos\alpha)^2 + (x-1)^2 - 2(2 + \cos\alpha)(x - 1)(\cos[180 - \alpha]) \quad (1)$$

Then $(x = 1)^2 \leq \phi^2$ can be combined with (1) and reduced to:

$$x \leq (4 - \cos^2\alpha)/(4 - 4\cos\alpha - 2\cos^2\alpha) \quad (2)$$

For $\alpha = 90°$, $x \leq 1$ which requires $1 \leq \phi - 1$ and $2 \leq \phi$

As α decreases, x increases, and therefore ϕ must increase. At about $\alpha = 43°$, the denominator in (2) becomes zero, and then becomes increasingly negative as α approaches zero. In this range where $0 < \alpha < \pm 43°$ since $x < 0$, then $0 < \phi - 1$. That is, in this range there is no constraint on the radius chosen for the ski-tip design.

A more complicated mathematical solution would involve segments of an Archimedean spiral at either end, rather than segments of a circle. This spiral is chosen because the change of radius of curvature is linear with respect to the change of angle of arc. Therefore, the absolute amount of change of radius along the inner mold is the same as the amount of change of radius along the outer mold arc of the same angle. In order for the tail end of the flipped mold-half to fit, the inner curve must again clear the outer curve. The constraints on the possible arcs or ranges of radii of curvature are beyond my analytic ability.

A final possibility circumvents the mathematical interest of the problem. Cut the required shape in any reasonable location across the plywood sheet using a one-inch diameter router bit. This will require a curve without any discontinuities in the rate of change of radius of curvature along the runner, and with a minimum one half inch radius of curvature at any point. The two mold halves are used in the position from which they are cut and there is no nonfunctional part of the mold.

Also solved by Bruce Brittain, Matthew Fountain, Winslow Hartford, Thomas Harriman, Doug Milliken, Fred Furland, Ken Rosato and the proposer.

FM 4. Charles Piper takes us from dogsled runners to the human variety with a problem he believes dates back to a circa 1850 book *Exercises in Algebra*, by Jones. *A* offers to run three laps while *B* does two but gets only 100 yards into his third lap when *B* wins. He then offers to run four laps to *B*'s three, and quickens his pace in the ratio of 4:3. *B* also quickens his pace, in the ratio of 9:8, but in the second lap falls off to his original pace, and in the third goes only 9 yards for the 10 he went in the first race. *A* wins the race by 180 yards. How long is each lap?

Unfortunately, as printed the problem has no solution. Harry Zaremba has impressed me again, this time by finding this very problem in "Puzzle Corner" a dozen years ago (it did not look familiar to me!) where it was stated correctly. The corrected version appears in the problem section above.

Also solved by Ken Rosato, Thomas Harriman, Randall Whitman, Fred Furland, Matthew Fountain, Linda Kalver, Winslow Hartford, Gordon Rice, Avi Ornstein, Joel Feil, and the proposer.

F/M 5. By analogy with palindrome, a validrome is a sentence, formula, relation or verse that remains valid whether read forwards or backwards. For example, relative to (prime) factorization, 341 is a factorably validromic number, since 341 = 11·31, and when read backwards gives 13·11 = 143, which is also correct. What is the largest factorably validromic number you can find? Remember that 1 is not prime so $n = n \cdot 1$ is not valid.

Peter Groot offers:

770700000012192010000001101 =
70000000001101·11010000000001.

The proposer, Albert Mullin, first defines R_n as the base 10 number consisting of *n* 1's. For example, R_3 = 111 Mullin then notes that R_{1031} has been shown to pass probabilistic tests that make it very likely that it is prime. Finally, Mullin offers as his answer

$$R_{1031} \cdot 101 = 1122 \cdots 2211$$

Robert High found some (smaller) solutions like the above but then added

What one really would like to find are those rare examples of non-palindromic numbers **none of the prime factors of which are themselves palindromic,** which are factorably valindromic. The *smallest* example of such I found is:

1469 = 13 × 113; 9641 = 31 × 311

some other examples are:

12769 = 113 × 113; 96721 = 311 × 311
13273 = 13 × 1021; 37231 = 31 × 1201

15899 = 13 × 1223; 99851 = 31 × 3221
115373 = 113 × 1021; 373511 = 311 × 1201
124639 = 113 × 1103; 936421 = 311 × 3011
135713 = 113 × 1021; 317531 = 311 × 1201
392899 = 13 × 30223; 998293 = 31 × 32203

998293 is the largest such "non-trivial" factorably valindromic number I found under 1,000,000. Note the special property of the pairs [113,311] and [1021,1201]: all four pairwise products 115373, 135713, 317531, and 373511 are factorably valindromic! It would be nice to find a set of four primes such that all *six* pairwise products were non-trivial factorably valindromic numbers, but that may be asking too much. I haven't found any non-trivial factorably valindromic numbers with three or more prime factors either, but I only searched up to 1,000,000. (Including what I call the trivial cases, I found a total of 995 factorably valindromic numbers under a million.)

Thomas Harriman agrees and writes:

Aesthetically each prime factor should be a different number when reversed, and the factors should be different from each other in either order—even though these conditions were not imposed in the original statement of the problem.

First of all, the largest primes discovered are of the form 2^p-1, where it's necessary but not sufficient that p be prime, and their only possible factors are of the form 2kp + 1 where k is a positive integer (Fermat, 1640). (The prime requirement and the factor limitation are what made the search for & testing of huge postulated primes feasible, and the record to date is for p = 19937.) But the likelihood that two such primes create the reverse of their product when they themselves are reversed is very small. Further, for the purposes of the problem, they must also be prime when reversed and there are no helpful formulas for limiting the search for reversible primes. Accordingly we're on our own.

It's obvious that to be reversibly prime the factors must begin and end with any of the digits 1, 3, 7, or 9. If what's in between is all zeroes then the product of two such factors will always be the reverse of the product of the reversed factors, provided that the four individual products of the beginning and ending digits are less than 10 and that the sum of the interior and exterior products are also. In other words, A00BxC00D = [AxC]x1000000 + [AxD + BxC]x1000 + [BxD] and similarly for the reverse: when each of the expressions in brackets results in a single digit, reversibility is satisfied. (Reversibility can occur by luck in other ways but not dependably.)

But all zeroes in the middle would leave us with two few numbers among which to find some primes. Use of 7 and or 9 as the beginning digit for one factor doesn't allow much flexibility for the internal digits even when 1's are used for the other three beginning & ending digits. On the other hand, factors of the form 3nnnnnnnn1 allow a reasonable quantity of 2's and 1's as well as 0's to be used as interior digits while maintaining reversibility. The range of possibilities is given by the following pairs of factors representing the upper ranges of values with reversibility.

(3322222221) (3001000001): central 1 may move to right
(3322221111) (3010001001): central 1's may move to right
(3321111111) (3011010001): same as above without bunching

As a practical matter four reversible primes of the lefthand type were found: 3322201021, 3322012021, 3322000001, and the largest given below, without invetigating numbers less than the third one. (This result conforms pretty well to the prime number theory that the quantity of primes up to a given number is equal to the natural log of the number.) Two of the righthand type were found without really trying: 3021000001 and the one given below.

The largest reversible product found with reversible prime factors, limited essentially by the power of a ten-digit TI Programmable 59, is:

(3322221011) (3010000001) = 9999885246432221011
(1101222233) (1000000103) = 1101222346425889999

Also solved by Ken Rosato, Harry Zaremba (who offers the conjecture that there is no largest factorably validromic number), Winslow Hartford, and Matthew Fountain.

Better Late Than Never

1988 M/J 2. Dave Mohr notes that in "correcting" his spelling of *The Compleat Strategyst* I actually converted it from correct to wrong. Although Mohr graciously assigned the blame to my automatic spelling checker, *mea culpa*.

1989 JAN 1. Lyndon Tracy and Hiroshi Yabe have responded.

Proposers' Solutions To Speed Problems

SD 1. 10,752. A finished 2 × 4 is 1.5 inches by 3.5 inches.

SD 2.