ALLAN J. GOTTLIEB, '67

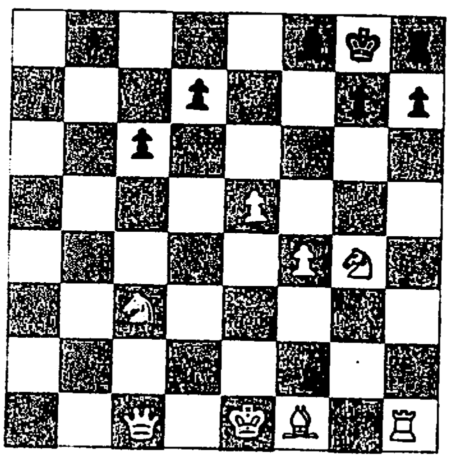
Lotteries and the Mnage Problem

Solving our JUL 2 problem led Jim Landau to notice that the radius of the inscribed sphere (to a tetrahedron) is 1/3 the radius of the circumscribing sphere. He conjectures that the radius of the inscribed n-dimensional sphere of an n dimensional regular tetrahedron is 1/n the radius of the circumscribing sphere, where the interesting part is to figure out what is meant by an n-dimensional tetrahedron.

To answer a question of Tony Merz, Peter Lax does indeed still teach at NYU and is now in addition director of the Courant Mathematics and Computing Laboratory.

Problems

N/D 1. We begin with a chess problem from Jim Landau in which White is to move and mate in two.



N/D 2. The following problem, which first appeared in the February 1986 issue of *IEEE Potentials*, is from James Rautio: Determine, without using a computer or calculator, which is larger, e^π or π^e.

N/D 3. Matthew Fountain wants you to show that there are no positive integer solutions to:

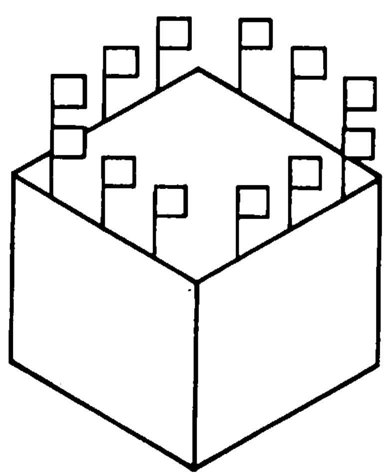
$X^3 + Y^3 + 4 = Z^3$

N/D 4. David DeWan, a lottery fan, wonders what is the probability that six random numbers chosen (without replacement) from 1 to 36 will have at least one adjacent pair (like 12 and 13, for example).

N/D 5. An anonymous Baker House alumnus wants you to solve the Mnage problem, where you have a certain number of couples to dinner and you wish to seat them at a round table, with men and women alternating and all the couples separated. How many different arrangements are possible for three couples? For four? How about ten?

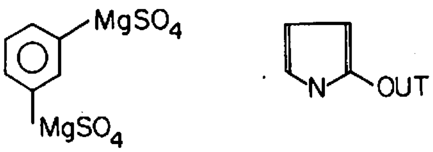
Speed Department

SD 1. Frank Rubin has a fortress laid out as a square building 100 yards on a side and 100 feet high. It stands in the middle of a flat plain. To identify the fort, the defenders want to place 3-foot by 6-foot flags on 12-foot poles around the perimeter of the (flat) roof in such a way that a person standing on the ground would see at least three flags from any direction. What is the smallest number of flags and poles required: Here is an example with 12 flags and poles.



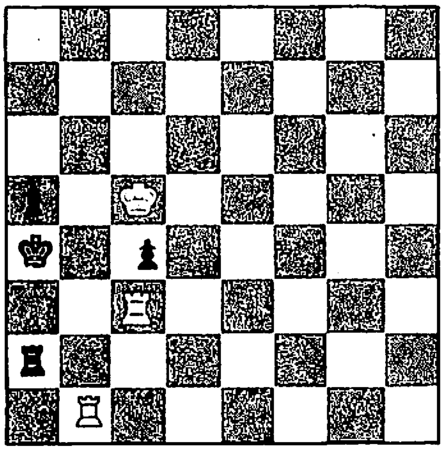
SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, THE COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y. 10012.

SD 2. Lester Ruth wants you to name the following compounds:



Solutions

JUL 1. White is to move and mate in three.

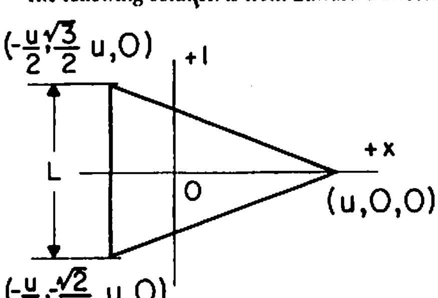


The following solution is from Mark Campbell: White's first move is Rb5. If Black responds with Ra3, White's play is Rxc4 mate. If Black responds with Ral, White's play is Rb2! Then if Black plays Ra2, White follows with RxR mate; or if Black plays Ra3, White's play is Rxc4 mate (as before). Or if Black responds to White's first move by putting R on any other square, rank 2, then:
2. Rax5 KxR
3. Ra3 mate.
Only the black rook can move; if it is sent to a3, White replies with the immediate mate, Rxc4. Black may think of Rb2; if then RxR, stalemate. But White has a reply for any move of the Black rook along the second rank—i.e.,
2. Rxa5 KxR
3. Ra3 mate.
Seeing this, Black may respond to White's first move with Ral, to thwart the Ra3 play. But White has an ingenious answer: Rb2. Now Black cannot got back to a2 because of the RxR mate. Ra3 is terminal, because it leads to Rxc4 mate. If the Black rook leaves the first file, White's response is Ra2 mate.

Also solved by Elliott Roberts, Harold Solomon, Jose Figueroa, Greg Spradlin, Matthew Fountain, Robert Johnson, Robert Bart, Ronald Raines, and the proposer, Matthew Ek.

JUL 2. Find the radius of a sphere circumscribing a regular tetrahedron.

The following solution is from Edward Dawson:



Using rectangular coordinates, let one face of the regular tetrahedron lie in the X − Y plane with vertices at (u,0,0), $(-u/2, \sqrt{3}u/2, 0)$, and $(-u/2, -\sqrt{3}u/2, 0)$. Let the fourth vertex be at (0,0,h) on the z-axis, and let the center of the circumscribing sphere be at (0,0,c). The edge length $L = \sqrt{3}u$, and the distance from vertex (u,0,0) to vertex (0,0,h) is $\sqrt{u^2 + h^2}$, so that
$L^2 = u^2 + h^2 = 3u^2$, and
$h = \sqrt{2}u = \sqrt{2/3}(L)$.
The distance of the center of the sphere from the vertex at (0,0,h) is equal to its distance from the vertex at (u,0,0). Therefore,
$h - c = \sqrt{u^2 + c^2}$, and
$c = (h^2 - u^2)/2h = u/2\sqrt{2}$.
The radius of the circumscribing sphere is
$h - c = \sqrt{2}u - u/2\sqrt{2} = 3u/2\sqrt{2} = \sqrt{3/8}(L)$.

Also solved by Avi Ornstein, Daniel Morgan, Dennis White, Harry Zaremba, Greg Spradlin, Jim Landau, John Chandler, Jose Figueroa, Kelly Woods, Ken Rosato, Matthew Fountain, Norman Wickstrand, Robert Johnson, Scott Berkenblit, Robert Bart, and Winslow Hartford.

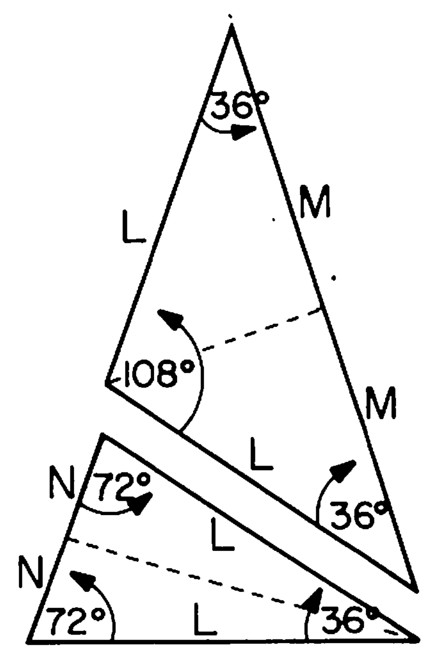
JUL 3. A (24-hour) digital watch has many times that are palindromic, such as 1:00:01, 23:22:32, :11, 2:44:42, etc. (ignore the colons). Find the two closest such times. Find the two that differ closest to 12 hours. Find the longest time span without a palindromic time.

Scott Berkenblit assumes that although ":11" is a valid palindromic time, ":01" is not (because the leading zero cannot be dropped). If this assumption is not made, :01 and :02 are consecutive palindromic times. Making the assumption gives a more interesting problem. The two closest palindromic times are 9:59:59 and 10:00:01; and 1:33:31 and 13:33:31 differ by exactly 12 hours. To determine the longest span without a palindrome, Mr. Berkenblit notes that times beginning with 16:, 17:, 18:, and 19: cannot be palindromic. The latest palindrome before 16:00:00 is 15:55:51, and the earliest one after 19:59:59 is 12:00:02, giving a span of 5:04:02.

Also solved by Avi Ornstein, Billy Eccles, Daniel Morgan, Harry Zaremba, John Chandler, Jose Figueroa, Ken Rosato, Matthew Fountain, Rik Anderson, Robert Johnson, Robert Bart, and the proposer, Nob Yoshigahara.

JUL 4. Find the exact value of (cos 36°) − (cos 72°).

Matthew Fountain sent us two solutions. The first is a cute geometric derivation; however Fountain admits that he figure this out only after the trigonometric solution revealed the answer.



Cos36° − cos72° = 1/2. In the upper isosceles triangle cos36° = M/L. In the lower isosceles triangle cos72° = N/L. Lowering the upper triangle down onto the lower triangle results in a larger isosceles triangle with equal sides of length L + 2N = 2M. Therefore cos36° − cos72° = M/L − N/L = (L/2 + N)/L − N/L = 1/2.

Fountain begins his trigonometric solution by noting

$\cos(2 \times 36°) = \sin(36°/2)$.

Substituting the identities

$\cos(2X) = 2\cos^2(X - 1)$ and

$\sin(X/2) = \sqrt{(1 - \cos X)/2}$ produces

$2\cos^2(36° - 1) = \sqrt{(1 - \cos 36°)/2}$.

Squaring both sides and collecting all terms on the left results in

$4\cos^4 36° - 4\cos^2 36° + (1/2)\cos 36° + 1/2 = 0$.

Dividing first by cos36° + 1 and next by cos36° − 1/2, neither of which equals zero, yields

$4\cos^2 36° - 2\cos 36° - 1 = 0$.

Noting that the first term equals 2cos72° + 2, we make this substitution and obtain

$2\cos 72° - 2\cos 36° + 1 = 0$,

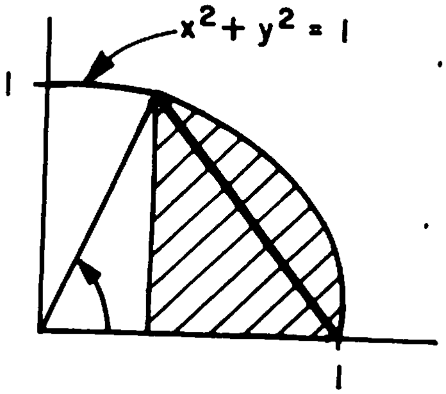
which is equivalent to cos36° − cos72° = 1/2.

Charlotte Helin adds her thanks "for all the fun I had with that innocent-looking problem." She writes, "After I solved the problem using geometry and one trigonometric identity, I suddenly realized that from the fact that cos36° − cos72° = 1/2, it would be possible to calculate the functions of 36° (using one trigonometric identity). So I did. I was not aware that this could be done for any angles except 30° and 45° (and their families). You remember, Hipparchus did that and made up a trig table in intervals of 7 1/2°. With a 'calculated' value for cos36°, I could produce a table in intervals of 3°. Because I was curious to see what it looked like, I calculated the value of sin 6°.

'Then, at some point when I was fiddling around with these numbers, I noticed *what the cos36° was!* $(1 + \sqrt{5})/4$ is one-half the golden ratio. So I looked phi up in one of Martin Gardner's books and found that it has been known for centuries that phi was the ratio of a side to the base of the isosceles triangle with a vertex angle of 36° and base angles of 72°. If I had been aware of this from the beginning, I could have solved for the sin 18°, then the cos36°, then the cos72° and done a little subtracting. And, of course, then I would have missed the fun of discovering all this stuff for myself. A little ignorance is sometimes an entertaining thing."

Also solved by Avi Ornstein, Daniel Morgan, Dennis White, Edward Dawson, Greg Spradlin, Farrel Powsner, Harold Solomon, Harry Zaremba, John Chandler, Jose Figueroa, Kelly Woods, Ken Rosato, Norman Spencer, Robert Johnson, Scott Berkenblit, Steven Feldman, Winslow Hartford, Robert Bart, and the proposer, Dennis Clougherty.

JUL 5. Let A be the area of the shaded curved region, and B the area of the triangle. Find $\lim_{\theta \to 0} A/B$.



The following solution is from John Chandler (via Arpanet and Bitnet):

Since the radius of the circle is 1, the area of the shaded region is $\theta/2 - \sin\theta\cos\theta/2$, and the area of the triangle within the shaded region is $(1 - \cos\theta)\sin\theta/2$. Since both areas vanish when θ goes to zero, we must apply l'Hopital's rule to find the limit of their ratio, i.e., take the limit of the ratio of their derivatives. The derivatives, $1/2 - \cos 2\theta/2$ and $[\cos\theta - \cos 2\theta]/2$ however, also vanish at θ = 0, so we must apply the rule again to get $\sin 2\theta$ and $\sin 2\theta - \sin\theta/2$, and a third time to get $2\cos 2\theta$ and $2\cos 2\theta - \cos\theta/2$. These quantities approach 2 and 3/2, respectively, for a limiting ratio of 4/3.

Also solved by Daniel Morgan, Dennis White, Edward Dawson, Harold Solomon, Harry Zaremba, Jose Figueroa, Kelly Woods, Ken Rosato, Matthew Fountain, Greg Spradlin, Robert Johnson, Scott Berkenblit Steven Feldman, Howard Stern, Robert Bart, and the proposer.

Better Late Than Never

JAN 1. Howard Stern, the proposer, notes that all solutions to date have been only asymptotic approximations and wonders if an exact solution exists.

F/M 1. Greg Spradlin found a shorter "Australian" solution.

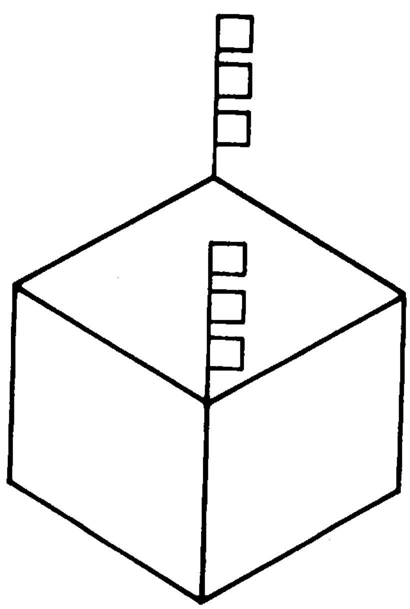
M/J 3. Naomi Markovitz has responded.

APR 2. Unfortunately Robert Bart's solution must have somehow become lost in transit. He had recognized that this problem is the classic "problem of Appolonius" and can be constructed with ruler and compass alone. There are eight exact solutions to the general case.

M/J 1, 2, 3, 4, and 5. Richard Hess has responded.

Proposers's Solutions to Speed Problems

SD 1.



SD 2. "Meta-physics" and "out on pyrrow".