ALLAN J. GOTTLIEB, '67

Seeking the Constants in the Physical Constants

Since this is the first issue of a new academic year, I once more review the ground rules under which this department is conducted.

In each issue I present five regular problems (the first of which is chess, bridge, or computer-related) and two "speed" problems. Readers are invited to submit solutions to the regular problems, and three issues later one submitted solution is printed for each problem; I also list other readers whose solutions were successful. For example, solutions to the problems you see below will appear in the February/March issue. Since I must submit that column sometime in November, you should send your solutions to me during the next few weeks. Late solutions, as well as comments on published solutions, are acknowledged in the section "Better Late Than Never" in subsequent issues.

For "speed" problems the procedure is quite different. Often whimsical, these problems should not be taken too seriously. If the proposer submits a solution with the problem, that solution appears at the end of the same column in which the problem is published. For example, the solutions to the issue's "speed" problems are given below. Only rarely are comments on "speed" problems published or acknowledged.

There is also an annual problem, published in the first issue of each new year; and sometimes I go back into history to republish problems that remained unsolved after their first appearances.

Problems

OCT 1. Lawrence Kells reports that a friend of his once held

♠ 4 3 2
♥ 4 3 2
♦ 4 3 2
♣ 5 4 3 2

SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, THE COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y. 10012.

Despite the lack of power, he took five tricks in his own hand. Nobody made any illegal plays. What was the deal and how did the play go?

OCT 2. Hy Tran wants you to show that for k, r, and positive integers the expression

$$\sum_{k=1}^{r-1} \frac{r!n^k}{k!(r-k)!}$$

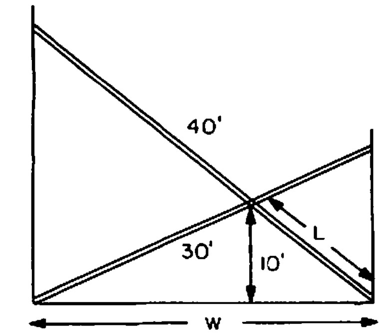
is always even—i.e., an integral multiple of 2.

OCT 3. The following problem first appeared in *Computers and People* in 1985. A "numble" is an arithmetical problem in which digits have been replaced by capital letters; there are two messages, one which can be read right away and a second one in the digit cipher. The problem is to solve for the digits. Each capital letter in the arithmetical problem stands for just one digit 0 to 9. A digit may be represented by more than one letter. The second message, expressed in numerical digits, is to be translated (using the same key) into letters so that it may be read; but the spelling may use puns or deliberate (but evident) misspellings, or may be otherwise irregular, to discourage cryptanalytic methods of deciphering.

```
          T H E
    ×   W O L F
    -----------
        K T L O
      E W T F
    S W E W
  H H W L
  -------------
= W L I W O O O
```

OCT 4. Here is a ladder problem from Joseph Molitoris and George Butwin. It looks familiar so I would not be surprised to hear that a similar if not identical problem appeared in *Puzzle Corner* ten or fifteen years ago:

Given an alley of width W. Two ladders of length 40 ft. and 30 ft. are laid against opposite walls. They intersect 10 ft. above the ground. Find W and L, the width and one of the lengths of intersection. (*See diagram next page.*)



OCT 5. Oren Cheyette has a problem concerning the first digit of various physical constants. The answer may surprise you. It surprised me:
Suppose you take a table of physical constants expressed in scientific notation (e.g., the speed of light is 2.998×10^8m/s) and construct a histogram of the first digits of their mantissas (e.g., 2, for the speed of light). In other words, count how many times each of the digits 1 to 9 occurs as the leading digit. What functions *a priori* do you expect to best fit this histogram? That is, for a random physical constant, what is the probability that its leading digit is n?

Speed Department

SD 1. Jim Landau wants to know why for any x between 0 and $\pi/2$ the sequence
x, cos(x), cos(cos(x)), cos(cos(cos(x))), . . .
converges to .739085 . . .

SD 2. Our last problem is from Harry Zaremba:
Let the decimal system numbers 31, 331, 3331, . . . be represented by the shorthand symbol 3_k1 in which the subscript k = 1, 2, 3, . . . signifies the number of the threes to the left of the one. What is the smallest value of k for which the number 3_k1 is composite? What are the number's prime factors?

Solutions

M/J 1. Construct a hypothetical bridge hand that contains the maximum possible number of finnesses, all of which would be successful. Consider spades as trump.

The following solution is from Winslow Hartford: I come up with 9 finesses. It's easy to plan the hand so N-S make either 7 spades or 7 no-trump. One entry is provided by the opening lead; the remaining 8 entries are readily available through winning finesses and 4 aces. The opening lead is immaterial.

♠ ---
♥ ----
♦ A Q 10
♣ A Q 10

♠ --
♥ ---
♦ K J 9 -
♣ K J 9 -

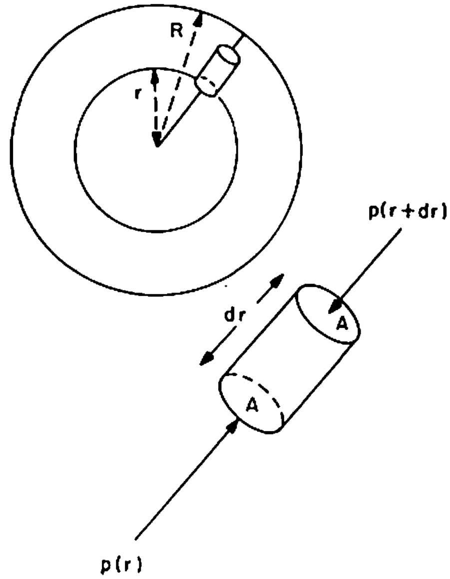
♠ K J 9 7
♥ K J 9
♦ ---
♣ ---

♠ A Q 10 8
♥ A Q 10
♦ ---
♣ ---

Also solved by the proposer Jim Landau, who reports that Allen Beadle, mentioned in the original statement of M/J 1, is also the (heretofore) anonymous slide rule expert who "computed" $.99^{100}$ mentally. Moreover, it was Beadle who told Landau the Zorn's Lemmon riddle.

M/J 2. What is the pressure due to gravity at the center of a spherical heavenly body of uniform density, mass M, and radius R?

The following solution is from Dennis White: Consider a small right cylinder with its axis placed radially within the spherical body:



The gravitational force on it is provided by the central sphere "underneath" with radius *r* (outer layers have a net zero attraction on it), and this sphere's mass can be considered to be concentrated at 0. Taking the cylinder to be a point mass, this force is
$Fg = -G[(4/3\pi r^3\rho)(Adr\rho)]/r^2 = -4(\pi G\rho^2 Ardr)/3$
where ρ is the body's density. For the cylinder to be at equilibrium, this must be matched by the buoyant force provided by the pressure differential $dp = p(r + dr) - p(r)$. (The pressure is a function of r alone by symmetry.) This force is
$F_b = Ap(r + dr) - Ap(r)$
$= A[p(r + dr) - p(r)]$
$= Adp$
Setting $F_g = F_b$, we have
$-4(\pi G\rho^2 Ardr)/3 = Adp$, or
$dp/dr = -4[(\pi G\rho^2)r]/3$.
Solved, with the boundary condition $p(R) = 0$, this gives
$p(r) = 2\pi G\rho^2(R^2 - r^2)/3$
or substituting
$\rho = M/(4/3)\pi R^3$
$p(r) = (3/8)G(M^2/\pi R^6)(R^2 - r^2)$,
and so p(0), the pressure at the center, is
$p(0) = (3/8)G(M^2/\pi R^6)(R^2 - r^2) = (3/8)G(M^2/\pi R^4)$.

Also solved by Matthew Fountain, Winslow Hartford, and the proposer, Bruce Calder.

M/J 3. Find nine single-digit numbers with sum 45 and product 362880. One solution is (1,2,3,4,5,6,7,8,9).

The following solution is from Bob Marshall, who says "thanks for publishing this puzzle that I could solve":
Start by observing that all such sets must be assembled from numbers included in the prime factorization of 362880 with the optional addition of one or more 1s. (Note that the starter solution included a 1; therefore this must be permitted.) The prime factorization of 362880:
$363880 = 2^7 \times 3^4 \times 5 \times 7$.
Note immediately that each solution must include the single digit numbers 5 and 7, because these factors must be included to achieve the product and any multiple of either of these numbers is not a single digit. The case of whether or not a 1 is included in the solution will be dealt with specially because the 1 does not change the product but will change the sum. For the case of no 1s the factors $2^7 \times 3^4$ must be distributed among 7 single digit numbers. These are constructed by inspection as follows:

Digits

1	2	3	4	5	6	7	8	9	Sum
9	9	8	2	2	2	2	5	7	46
9	9	4	4	2	2	2	5	7	44
9	8	6	3	2	2	2	5	7	44
9	8	4	3	3	2	2	5	7	43
9	6	6	4	2	2	2	5	7	43
9	6	4	4	3	2	2	5	7	42
9	4	4	4	3	3	2	5	7	41
8	8	3	3	3	3	2	5	7	42
8	6	6	3	3	2	2	5	7	42
8	6	4	3	3	3	2	5	7	41
8	4	4	3	3	3	3	5	7	42
6	6	6	6	2	2	2	5	7	42
6	6	6	4	3	2	2	5	7	41
6	6	4	4	3	3	2	5	7	40
6	4	4	4	3	3	2	5	7	39

As can be seen from the Sum column none of these sets of single-digit numbers has the required sum. For the case with a single 1 the factors $2^7 \times 3^4$ must be distributed among six single-digit numbers. These are constructed by inspection as follows:

Digits

1	2	3	4	5	6	7	8	9	Sum
9	9	8	4	2	2	1	5	7	47
9	9	4	4	4	2	1	5	7	45
9	8	8	3	3	2	1	5	7	46
9	8	6	6	2	2	1	5	7	46
9	8	6	4	3	2	1	5	7	45
9	8	4	4	3	3	1	5	7	44
9	6	6	4	4	2	1	5	7	44
9	6	4	4	4	3	1	5	7	43
8	8	6	3	3	3	1	5	7	44
8	6	6	4	3	3	1	5	7	43
8	6	6	6	3	2	1	5	7	44
6	6	6	6	4	2	1	5	7	43
6	6	6	4	4	3	1	5	7	42

In this case there are two solutions—the combinations: 1, 2, 4, 4, 4, 5, 7, 9 and 9; and 1, 2, 3, 4, 5, 6, 7, 8 and 9. This latter solution is the "given" solution and my method would be proved nonexhaustive if it failed to find this solution. For the case with two 1s, the factors $2^7 \times 3^4$ must be distributed among five single-digit numbers. These are constructed by inspection as follows:

Digits

1	2	3	4	5	6	7	8	9	Sum
9	9	8	8	2	1	1	5	7	50
9	9	8	4	4	1	1	5	7	48
9	8	8	6	3	1	1	5	7	48
9	8	6	6	4	1	1	5	7	47
8	6	6	6	6	1	1	5	7	46

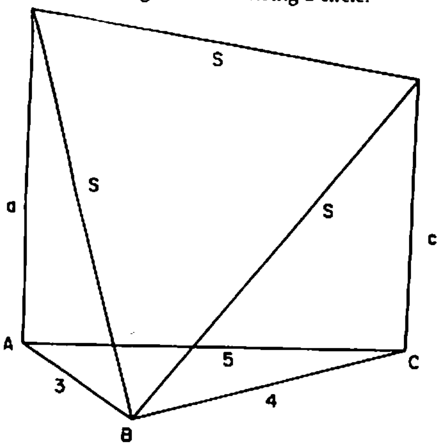
Thus there are no solutions for the case of two 1s. For the case with three 1s the factors $2^7 \times 3^4$ must be distributed among four single-digit numbers. However, this cannot be done as any combination of those factors distributed among four numbers yields at least one two-digit number. Therefore the search for solutions is complete. To recap, the two solutions are:
1 2 3 4 5 6 7 8 9
1 2 4 4 4 5 7 9 9

Also solved by Thomas Stowe, Avi Ornstein, Mike Gennert, Michael Jung, Alan Taylor, Dick Robnett, James Landau, John Woolston, Larry Bell, Matthew Fountain, Winslow Hartford, Ken Rosato, and Steve Feldman.

M/J 4. Find the axes of the largest (in area) ellipse that can be inscribed in a triangle having sides of length 3, 4, and 5 inches.

The following solution is from the proposer, Matthew Fountain:

The major and minor axes are 2.93986 and 1.57110. When the 3,4,5 triangle is viewed from an angle that makes it appear to be equilateral, the maximum area ellipse must appear as a circle. The foreshortening takes place in the direction of the major axis of the ellipse, with no foreshortening in the direction of the minor axis, so it may be concluded that the angle of view does not affect the ratio of apparent area of ellipse to apparent area of triangle. If an ellipse does not appear as a circle when the triangle appears equilateral, it follows that if the ellipse is viewed as a circle, the triangle would then appear not an equilateral triangle, implying that the ratio of ellipse area to triangle area was smaller than the previous case. An equilateral triangle is the smallest triangle circumscribing a circle.



Similarly, a proper-size equilateral triangle can be foreshortened into a 3,4,5 triangle. The figure above shows such an equilateral triangle with sides of length S, tilted so that its projection on a plane is the 3,4,5 triangle ABC. One vertex is the distance a above A, one vertex is the distance c above C, and the remaining vertex lies in the plane at B. S is found by writing $a^2 = S^2 - 3^2$, $c^2 = S^2 - 4^2$, and $(a-c)^2 = S^2 - 5^2$ and solving for S. The elimination of a and c results in the equation

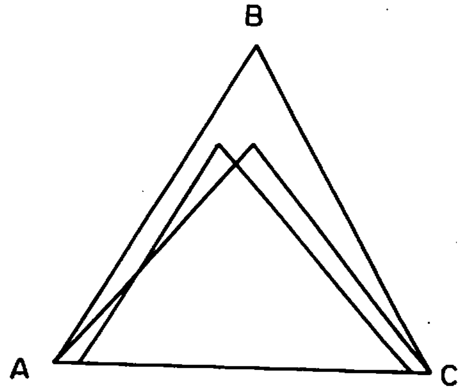
$3S^4 + 100S^2 + 24^2 = 0$,

with the solution S = 5.091984. The altitude of the equilateral triangle is $(S/2)\sqrt{3} = 4.409787$ and the diameter of the inscribed circle is two-thirds the altitude—that is, 2.939858. This is also the major axis of the ellipse in the 3,4,5 triangle. The respective areas of the equilateral triangle and the 3,4,5 triangle are (5.091984)(4.409787)/2 = 11.227282 and (3)(4)/2 = 6. The minor axis of the ellipse is (2.9398858)(6)/11.227282 = 1.571096.

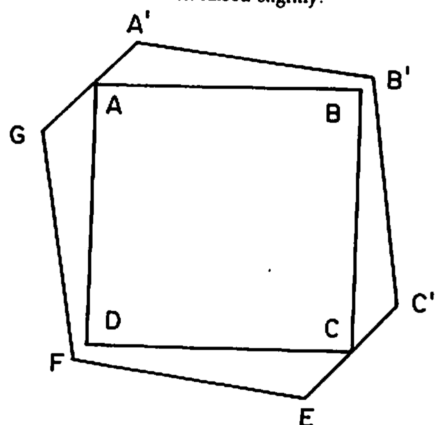
Also solved by Norman Wickstrand, Winslow Hartford, Edward Dawson, and Dennis White.

M/J 5. Through which regular polyhedra can one :arve a hole such that another regular polyhedron of the same size and type can pass? For example, :learly a cube with unit edge can pass through a uitably cut hole in another cube with unit edge.

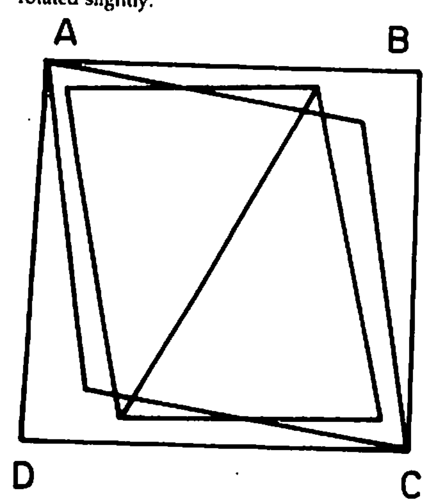
The following solution is from Matthew Fountain: he regular tetrahedron, the cube, and the regular ctahedron can each be passed through a hole in a olyhedron identical to it. The dodecahedron and :osahedron cannot. One procedure to determine ·hen this is possible is to view each polyhedron om various angles and compare silhouettes. When ne silhouette can fit into another, it is possible. If e polyhedron is rotated, it is possible to obtain oth silhouettes as projections of the polyhedron n one plane, in positions that assist comparisons :tween them. As a regular polyhedron rotates, its :rtices move in the surface of a sphere, their projections on a plane moving to and from the circumference of a circle. The dodecahedron and osahedron have so many vertices that no matter hat axis of revolution is chosen, the vertices of eir silhouettes are never going outward to the cle at the same time. This would be necessary for e silhouette to fit within another, as there is not ance for any appreciable rotation of one silhouette owing it to fit into another.



The figure above shows equilateral triangle ABC as the silhouette of the regular tetrahedron projected upon a plane parallel to one face. Tilting the tetrahedron backward shortens its silhouette as shown. Then rotating this tilted tetrahedron about a vertical axis through the center of its bottom edge shortens the bottom edge of the silhouette. The figure exaggerates the foreshortening that occurs in order to show it more clearly. The last silhouette will fit in ABC when raised slightly.



The second figure (above) shows the square ABCD as the silhouette of a cube projected upon a plane parallel to one face. Rotating the cube about its center on an axis parallel to AC produces the silhouette AA'B'C'EFG. As EF and GH are perpendicular to AC, the first silhouette will fit in the second when rotated slightly.



The third figure (above) shows the square ABCD as the silhouette of a regular octahedron projected upon a plane parallel to four edges. Rotating the octahedron about its center on an axis parallel to AC produces the diamond-shaped silhouette extending between A and C. An additional rotation of the octahedron about its center on an axis parallel to BD produces a silhouette lying within the ABCD.

Also solved by Winslow Hartford.

Better Late Than Never

JAN SD1. Sidney Williams has responded.

JAN 3. Dennis White notes that an unstated requirement was that the i^{th} term uses the digit i.

F/M 2, 3, 4. Winslow Hartford has responded.

APR 3. Joseph Muskat, Naomi Markovitz, and Winslow Hartford have responded.

APR 4. Naomi Markovitz and Winslow Hartford have responded.

APR 5. Joseph Muskat and Winslow Hartford have responded.

M/J SD1. Linda Kalver has responded.

M/J SD2. Both Turner Gilman and the proposer Jim Landau assert that the answer should be 4 meters and not 2.8284 . . . meters. Mr. Gilman writes: "The answer given would be correct if the 'lateen rig' was constructed logically, in which case the center of area of the triangular sail would be at the mast location. This would minimize rotational forces. But that is *not* the way the puzzle is presented. I 'assume that the lateen yard is attached to its center to a mast 2 meters high.' The key is the word *center*. There are some details omitted, i.e. the clearance between deck and sail and any excess height above the sail, but on the same basis that the proposer gets his 2.8284 meters I arrive at a solution of 4.0000 . . . meters!"

Proposers' Solutions to Speed Problems

SD 1. You are just using an iterative method to find the solution to cos(x) = x.

SD 2. All the numbers 3_k1 are prime for k equal to the consecutive numbers 1 to 7, inclusive. The smallest value of k for which the number is composite is 8:
3_81 = 333,333,331 = 17 × 19,607,843.