ALLAN J. GOTTLIEB

Stacked Deck and Houdini Polyhedra

Warren Seamans, director of the M.I.T. Museum, notes that the puzzle exhibition mentioned in the Feb/Mar issue will be showing at the Hudson River Museum in Yonkers, N.Y. from 6 April through 27 June.

Since it has been a year since I reviewed the criteria used to select solutions for publication, let me do so now.

As responses to problems arrive, they are simply put together in neat piles, with no regard to their date of arrival or postmark. When it is time for me to write the column in which solutions are to appear, I first weed out erroneous and illegible responses. For difficult problems, this may be enough; the most publishable solution becomes obvious. Usually, however, many responses still remain. I next try to select a solution that supplies an appropriate amount of detail and that includes a minimal number of characters that are hard to set in type. A particularly elegant solution is, of course, preferred. I favor contributions from correspondents whose solutions have not previously appeared, as well as solutions that are neatly written or typed, since the latter produce fewer typesetting errors.

Problems

M/J 1. Jim Landau writes that a friend of his, Allen Beadle, was in a bridge game with a certain timid (but nevertheless highly conceited) bridge player who refused to play finesses. Deciding that this player should be taught a lesson, Allen slipped in a stacked deck. The conceited player found himself holding a hand that contained the maximum possible number of finesses, all of which would be successful. You are to reconstruct the hand assuming spades are trump.

SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, THE COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y. 10012.

M/J 2. Bruce Calder wants to know the pressure due to gravity at the center of a spherical heavenly body of uniform density, mass M, and radius R.

M/J 3. Nob Yoshigahara has nine single-digit numbers with sum 45 and product 362880. What are they? One solution is (1,2,3,4,5,6,7,8,9). Can you find any other?

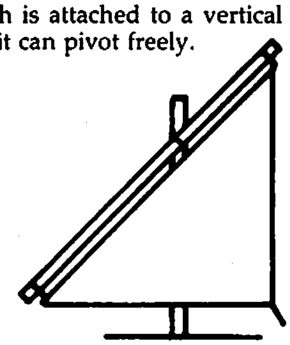
M/J 4. Matthew Fountain asks for the axes of the largest (in area) ellipse that can be inscribed in a triangle having sides of length 3, 4, and 5 inches.

M/J 5. Albert Mullin entitles the following problem *On Houdini Polyhedra*: Through which regular polyhedra can one carve a hole such that another regular polyhedron of the same size and type can pass? For example, clearly a cube with unit edge can pass through a suitably cut hole in another cube with unit edge.

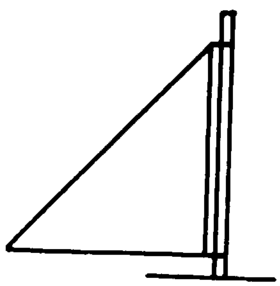
Speed Department

SD 1. Edward Lynch thinks that the sum of the integers from 1 to x is $x^2/2$. What is the smallest integral value of x (greater than 1) such Mr. Lynch's error is within 10 percent? Within 1 percent?

SD 2. Jim Landau claims to have canvassed his office looking for speed problems and found this one from George Claxton. On a boat with a "lateen rig" the sail is attached to a "lateen yard," which is attached to a vertical mast so that it can pivot freely.



On a "Marconi-rigged" boat, the sail is attached directly to the vertical mast.



Both rigs use triangular sails. One advantage of the lateen rig is that you can hang the same sized sail on a shorter mast. Let the sail be in the shape of an isosceles right triangle and assume that :he lateen yard is attached at its center :o a mast 2 meters high. How tall a mast vould be needed on a Marconi-rigged)oat for the same sail?

;olutions

\N 1. Find the minimum number of high card oints needed by a bridge partnership in order to isure making a 7NT contract.

For this problem the challenge seemed to be how ı interpret the problem. In particular, for a given ɔint count, are we to find the most favorable or ast favorable card placement? Ron Adelsman und a hand where the partnership has all 40 ɔints but cannot even make 6NT:

♠ A K
♥ A K
♦ A K Q
♣ A K Q J 4 3

10 9 8 7 6 5 4
10 9 8 7 6 5
—
—

♠ —
♥ 4
♦ 10 9 8 7 6 5
♣ 10 9 8 7 6 5

♠ Q J 3 2
♥ Q J 3 2
♦ J 4 3 2
♣ 2

.n Woolston found a hand where the partnership ; only 11 points and yet makes 7NT against any ense:

♠ A
♥ A Q J 10 9 8 7 6 5 4 3 2
♦ —
♣ —

ꞔ Q J 10 9 8 7 6 5 4 3 2
ꞔ
—
—

♠ —
♥ —
♦ A K Q J 10 9 8
♣ A K Q J 10 9

♠ —
♥ —
♦ —
♣ —

Jim Landau notes that if you can distribute the cards at will for the partnership but not for the opponents, then 19 points are required. This can be done by taking the previous example and giving North the other two aces.

Also solved by John Woolston, Matthew Fountain, Michael Bushnell, Ron Adelman, Steve Strauss, Turner Gilman, Bill Huntington, Winslow Hartford, and the proposer, John Boynton.

JAN 2. Given a regular tetrahedron whose edges are L units in length. What is the value of L if the number of units of volume of the tedrahedron is equal to the sum of the number of units of length of all edges?

The volume of a pyramid is 1/3 × base × height. For a regular tetrahedron, the faces are all equilateral triangles, so the area of the base is $(L^2\sqrt{3})/4$, where L is the length of an edge. To find the height, a cross-section is taken of the tetrahedron. The base of the triangle thus formed is $(L\sqrt{3})/2$, which is divided into two parts by the altitude, $(L\sqrt{3})/6$ and $(L\sqrt{3})/3$. The corresponding sides of the triangle are $(L\sqrt{3})/2$ and L. The Pythagorean Theorem gives the altitude of $(L\sqrt{6})/3$. This gives the tetrahedron a volume of $(L^3\sqrt{2})/12$. When this is set equal to the sum of the length of all the edges (6L), L is found to equal $6\sqrt[4]{2}$ or 7.13524269...

Also solved by Alan Taylor, Arthur Kaplan, Avi Ornstein, Jim Rutledge, John Woolston, Ken Rosato, Mary Lindenberg, Ray Kinsley, Steve Feldman, Matthew Fountain, Jim Landau, Winslow Hartford, and the proposer, Phelps Meaker.

JAN 3. Find the next term in the sequence 11,2,3,41,5,61,7,83,...

The ith term is the smallest prime using only digits from 1 to i. If, like Jim Rutledge, you require the ith term to *start* with the digit i, then the answer is 97. If, like Matthew Fountain, you do not make this requirement, then the answer is 19.

Yet another possibility exists as evidenced by Roger Milkman's solution; he writes from the University of Iowa:

I was astonished to learn of the existence of another analyst of the history of "scholar athletes." Walter Nissen is obviously referring to those Most Valuable Players on Iowa football teams whose numbers corresponded exactly with the number to times they attended class. This occurred nine times between 1907 and 1979, though not in the order given. The missing number is 97, whose name is lost in his academic exploits. He was named to the Big Ten all-academic team, did research on *recA* (a gene important in recombination in *E. coli*), and got a degree in theology. These last two achievements together account for his cognomen, recA theol. 97. The fact that this question was even proposed suggests that there are others who will know its answer, but who else will have a mind sufficiently recondite to note that the sequence also contains the smallest prime beginning with each of the digits 1 through 8 (and now 9) in ascending order?

Also solved by Roger Milkman, Steve Feldman, Arthur Kaplan, John Upton, Steve Reed, Jim Landau, Winslow Hartford, and the proposer, Walter Nissen.

JAN 4. Your doctor gives you n tablets, which you keep in a bottle. Each day you are supposed to take one-half tablet. Your method for doing so is to shake the bottle until either a full tablet or a half tablet falls out (assume the full and half tablets have the property that the probability one falls out is the proportion of that type in the bottle). If a half tablet falls out you take it. If a full tablet falls out, you break it in two, take one half and put the other half back in the bottle. If you keep track of the days that you break a tablet this will be a sequence of n numbers (always starting with a 1 since you only have full tablets when you start). Concentrate now on the *last* day you break a tablet. The earliest last day could be n and the last could be 2n − 1. All other days in between are possible and some occur more frequently than others. The question is: What is the average last day you break a tablet?

The following solution is from Matthew Fountain: The average last day is 2n − D where D = $\log_e n$ + 0.6. D also represents the average number of half tablets left after the last tablet is broken. I arrived at this value by first applying the Monte Carlo method. I wrote a short Pascal program, which set x: = n, y: = 0, and then performed WHILE x≥ 1 DO IF RANDOM ≤ x/(x + y) THEN BEGIN x: = x − 1; y: = y + 1; END ELSE y: = y − 1. Here x is the number of whole tablets and y is the number of half tablets in the bottle, and x/(x + y) represents the probability of shaking out a whole tablet. The program ends when x reaches 0 and y is number of half tablets remaining. The average values of y obtained on 200 executions each of the program for various values of n are listed below.

n	10	20	40	80	160	320	640	1280
y	2.89	3.72	4.44	4.96	5.44	6.29	7.00	7.40
Log(y)	2.30	3.00	3.69	4.38	5.08	5.77	6.46	7.16

I was then curious if the following method would confirm these values. Each day on the average x decreases x/(x + y) units while y decreases [y/(x + y)] − x/(x + y) units. This change approximates the differential equation dy/dx = (x − y)/x which has the solution y = kx − xlogx. With the condition y = 0 at x = n, y = xlogn − xlogx = xlog(n/x). Because x is the denominator of one side of the differential equation, the equation loses validity near x = 0, x being somewhat of an approximation. But it does show that the above values are reasonable, why D lies close to logx, and how D can be estimated for huge n. For example, for n = 1280 and x = 10, y = 10(log1280) − 10(log10) = 10(log128) = 48.5. I then started my Pascal program with the initial value of 10 whole tablets and 49 half tablets. The average of 200 runs was y = 6.96, close to the value of 7.40 obtained previously for n = 1280. Incidently, the solution to the differential equation is y = log(n) when x = 1, so that empirically the solution can be used to obtain approximate D by assuming x = 1.

It occurred to me that the average outcome D of a jar depended upon the outcome expected of two jars, one containing one less half pill, one containing one more half pill and one less whole pill. D would be the weighted average of the D's for the latter two jars, with the weighting dependent upon the proportion of whole pills to half pills in the original jar. It should be possible to start with simple cases and build up to any desired case. The diagram below shows how I obtained D's for jars containing a total of less than five whole and half pills.

Whole pills in jar	*Half pills in jar* 0	1	2	3	4
0	—	1.000	2.000	3.000	4.000
1	1.000	1.500	2.000	2.500	
2	1.500	1.083	2.167		
3	1.833	2.083			
4	2.083				

The cells of this array may be represented by V[x,y]. The top row cells V[0,y] contain the possible final outcomes. Each remaining cell contains the average number of half pills expected to result from a jar with contents defined by x,y. Thus V[1,1] contains 1 whole and 1 half pills. V[1,1] = 1.5 because there is an even chance that the next day the jar will be at either V[1,0], and ending up with an average of 1 half pill, or at V[0,2], and ending up with an average of 2 half pills. V[1,2] = 2.000 because a jar with 2 half pills and 1 whole pill is twice as likely to move to V[1,1] = 1.500 as it is to V[0,3] = 3.000. The calculation is (2*1.500 + 1*3.000)/(2 + 1) = 2.000.

Then I wrote a computer program that calculated similar arrays by repeated calculations of the relationship V[x,y] = (xV[x − 1,y] + yV[x,y − 1])/(x + y). To conserve memory, the program stored all the rows in the same one-dimensional array V[y]. This meant that as each new row was filled in it overwrote the part of the old row that was no longer required. Values of V[x,0] were printed out when calculated. The program obtained the following results. Note that |y − logy − 0.60| does not exceed 0.03 in any column.

n	10	20	40	80	160	320	640	1280
y	2.93	3.60	4.28	4.97	5.66	6.35	7.04	7.73
Log(y)	2.30	3.00	3.69	4.38	5.08	5.77	6.46	7.15

Also solved by Winslow Hartford, John Pullen Mary Lindenberg, and Matthew Fountain.

Better Late Than Never

Y1986. Several corrections and improvements have been found by Alan Foonberg, John Drumheller, Daniel Karlan, Alan Katzenstein, and Al Weiss.

4	$6-18/9$
9	$1^{86}\times 9$
22	$9+8+6-1$
58	$68-1-9$
62	$61+9-8$
64	$(18/9)^6$
70	$69+1^8$
72	$18+9\times 6$
78	$96-18$
87	$19+68$

1986 OCT 2. Larry Bell has responded.

OCT 3. Naomi Markovitz has responded.

N/D 1. Steve Strauss has responded.

N/D 2. Greg Spradlin, Larry Bell, and Naomi Markovitz have responded.

N/D 3. John Upton, John Cushnie, Greg Spradlin, and Steve Strauss have responded.

N/D 5. In addition to submitting a solution, John Upton suggests two variations to the towers of Hanoi problem. For both variations we keep the rule that no disk may be placed upon a smaller one. In the first variation, disk 1 (the smallest) many not be placed on 4, 2 cannot go on 5, etc. For the second variation disk 1 may move as normal or by exchanging with a disk on another pile. All other disks may only move by exchange.

1987 JAN SD1. The proposer, Jim Landau, noticed that we omitted the operation exponentiation from the list of permissible operators. This, however, did not cause readers much trouble. More serious was the loss of superscripted superscripts in the phototypsetting process. Specifically $7^{8^{91}}$ was printed as 7^{891} and $10^{10^{82}}$ was printed as 10^{1082}.

Responses were also received from Daniel Karlan, Greg Spradlin, Jim Rutledge, Al Weiss, John Baxter, John Woolston, Michael Bushnell, Steve Feldman, Steve Strauss, Wayles Browne, and Turner Gilman.

Proposers' Solutions to Speed Problems

SD 1. 9. 99.

SD 2. 2.8284... meters.