ALLAN J. GOTTLIEB

To Win Four Out of Seven, Must You Win the Fifth?

This being the first issue of another year, we again offer a "yearly problem" in which you are to express small integers in terms of the digits of the new year (1, 9, 8, and 7) and the arithmetic operators. The problem is formally stated in the "Problems" section, and the solution to the 1986 yearly problem is in the "Solutions" section.

Last night the New York Mets won the World Series after losing the fifth game, a timely introduction to A/S 4, the solution to which appears below.

Finally, Mary Lindenberg writes that a friend visiting Hong Kong met an M.I.T. alumnus living there who reads *Puzzle Corner* and asked about her. Ms. Lindenberg would like to correspond with this alumnus but does not know his name. I will be happy to serve as a go-between.

Problems

Y1987. Form as many as possible of the integers from 1 to 100 using the digits 1, 9, 8, and 7 exactly once each and the operators +, −, × (multiplication), / (division), and exponentiation. We desire solutions containing the minimum number of operators; and, among solutions having a given number of operators, those using the digits in the order 1, 9, 8, and 7 are preferred. Parentheses may be used for grouping; they do not count as operators. A leading minus sign *does* count as an operator.

JAN 1. John Boynton wonders what is the minimum number of high card points needed by a bridge partnership in order to insure making a 7NT contract.

JAN 2. Phelps Meaker has a regular tetrahedron whose edges are L units in length. What is the value of L if the number of units of volume of the tetrahedron is equal to the sum of the number of units of length of all the edges?

SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, THE COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y. 10012.

JAN 3. Walter Nissen wants you to find the next term in the sequence 11, 2, 3, 41, 5, 61, 7, 83, . . .

JAN 4. Our final problem is from Howard Stern, who writes:

Your doctor gives you n tablets, which you keep in a bottle. Each day you are supposed to take one half a tablet. Your method for doing so is to shake the bottle until either a full tablet or a half tablet falls out (assume the full and half tablets have the property that the probability one falls out is the proportion of that type in the bottle). If a half tablet falls out you take it. If a full tablet falls out you break it in two, take one half and put the other half back in the bottle. If you keep track of the days that you break a tablet this will be a sequence of n numbers (always starting with a 1 since you only have full tablets when you start). Concentrate now on the last day you break a tablet. The earliest last day could be n and the last could be $2n-1$. All other days in between are possible and some occur more frequently than others. The question is: What is the average last day you break a tablet?

Speed Department

SD1. Our first quickie, from Jim Landau, is related to Y1987:

What is the largest number that can be created with the digits 1, 9, 8, and 7 and the operations of addition, subtraction, multiplication, and division? What is the smallest non-negative number?

SD 2. Phelps Meaker's watch tells the 7 days of the week and 31 dates. Each moves ahead every 24 hours. But the mechanism that allows correcting the date for months of 28, 29, and 30 days does not work. On February 28, 1986 the watch says 14. When will the date be in sync with the calendar? For how long?

Y 1986. Use the digits 1,9,8, and 6 to form the integers from 1 to 100; the rules are the same as those given for Y1987, above.

The following solution is from George Aronson:

1. 1^{986}
2. $1 + (9-8)^6$
3. $9 \times 6/18$
4. $6 - 18/19$
5. $91 - 86$
6. $1^{98} \times 6$
7. $1^{98} + 6$
8. $9 - 1^{86}$
9. $1^{86 \times 9}$
10. $1^{86} + 9$
11. $(8 \times 9) - 61$
12. $81 - 69$
13. $96/8 + 1$
14. $98/(1 + 6)$
15. $96 - 81$
16. $16 \times (9 - 8)$
17. $19 - 8 + 6$
18. $9 \times 16/8$
19. $1 + 9 \times (8 - 6)$
20. $(1 + 9) \times (8 - 6)$
21. $19 + 8 - 6$
22. $196/8$
23. $91 - 68$
24. $1 + 9 + 8 + 6$
25. $1 + 8 \times (9 - 6)$
26. ——
27. $9 \times 18/6$
28. $89 - 61$
29. $8 \times 6 - 19$
30. ——
31. $8 \times (6 - 1) - 9$
32. $8 \times (1 + 9 - 6)$
33. $198 / 6$
34. ——
35. ——
36. $6^{(18/9)}$
37. $98 - 61$
38. $19 \times (8 - 6)$
39. $(8 \times 6) - 9^1$
40. $1 - 9 + (8 \times 6)$
41. ——
42. $(8 - 1^9) \times 6$
43. $91 - 6 \times 8$
44. $61 - 8 - 9$
45. $9 \times (6 - 1^8)$
46. $6 \times 9 \times 1 - 8$
47. $8 \times 6 - 1^9$
48. $1^9 \times 8 \times 6$
49. $68 - 19$
50. ——
51. $69 - 18$
52. $6 \times (1 + 9) - 8$
53. $9 \times (6 - 1) + 8$
54. $6 \times 81/9$
55. $9 \times (1 + 6) - 8$
56. $(9 \times 8) - 16$
57. $1 \times 9 + 8 \times 6$
58. $(1 + 9) + (8 \times 6)$
59. $68 - 9^1$
60. $68 + 1 - 9$
61. $61 \times (9 - 8)$
62. $62 + 9 - 8$
63. $(8 - 1^6) \times 9$
64. $(1 + 9 - 8)^6$
65. $8 \times (6 + 1) + 9$
66. $(19 - 8) \times 6$
67. $86 - 19$
68. $1^9 \times 68$
69. $1^9 + 68$
70. $(9 - 1) \times 8 + 6$
71. $9 \times 8 - 1^6$
72. $1^6 \times 9 \times 8$
73. $89 - 16$
74. $(1 + 9) \times 8 - 6$
75. $(1 + 8) \times 9 - 6$
76. $69 + 8 - 1$
77. $91 - 8 - 6$
78. $(1 \times 9 \times 8) + 6$
79. $1 + 9 \times 8 + 6$
80. $8 \times (1^6 + 9)$
81. $9^{16/8}$
82. $98 - 16$
83. $89 - 6^1$
84. $1 + 89 - 6$
85. $86 - 1^9$
86. $1^9 \times 86$
87. $1^9 + 86$
88. $89 - 1^6$
89. 89×1^6
90. $1^6 + 89$
91. $98 - 6 - 1$
92. $1 \times 98 - 6$
93. $1 + 98 - 6$
94. $86 + 9 - 1$
95. $1 \times (9 + 86)$
96. $1 + 9 + 86$
97. $98 - 1^6$
98. 98×1^6
99. $98 + 1^6$
100. $(1 + 9)^{8-6}$

Also solved by David Simen, Steve Feldman, Glen Rowsam, Randall Whitman, Harry Zaremba, Thomas Weiss, Joseph Feil, Peter Silvenberg, Richard Williams, Jim Rutledge, Jim Landau, Mark Johnson, Arthur Gelb, Alan Katzenstein, Avi Ornstein, R. Stephen Callaghan, Greg Spradlin, Eli Passow, W. Compton, A. Holt, and Allen Tracht.

A/S 1. What is of particular interest about the factors of the following numbers: 1271, 42477, 74989, 128929, 923521, 4424351, 4782969, 536215711, 2889101203, 98695877281, 424777960767, 1470848491213, 7532627125087, 7617609926757, 12893722612807, 17037029794091, 28917102427847, 170396299851737, 1703971665820979.

This computer-related problem was solved by Lawrence Bell, whose Z-100 PC has also calculated (so far) the first 2000 digits of π. Mr. Bell sent me his calculation with the suggestion that such digits have an affinity for the circular file. His solution to A/S 1 follows.

To solve this problem, I first wrote a straightforward BASIC program to find the factors. This didn't work at all for the larger numbers because of round-off error. In my next program, each number to be factored was treated as an array with each digit being one array element. An algorithm equivalent to long division was used to operate on the array one digit at a time. For each number, an ordered list of its factors (from smallest to largest) recreates the digits of π, appropriately truncated.

$\pi = 3.14159265358979323846 1.$

1. $1271 = 31 \times 41$
2. $42477 = 3 \times 14159$
3. $74989 = 31 \times 41 \times 59$
4. $128929 = 31 \times 4159$
5. $923521 = 31^4$
6. $4424351 = 31 \times 41 \times 59^2$
7. $4782969 = 3^{14}$
8. $536215711 = 31 \times 4159^2$
9. $2889101203 = 31 \times 41 \times 59^2 \times 653$
10. $98695877281 = 314159^2$
11. $424777960767 = 3 \times 141592653589$
12. $1470848491213 = 31^4 \times 1592653$
13. $7532627125087 = 31 \times 41 \times 5926535897$
14. $7617609926757 = 3^{14} \times 1592653$
15. $12893722612807 = 31 \times 415926535897$
16. $17037029794091 = 31 \times 41 \times 59^2 \times 653 \times 5897$
17. $28917102427847 = 31 \times 41 \times 59^2 \times 6535897$
18. $170396299851735 = 31 \times 41 \times 59^2 \times 653 \times 58979$
19. $1703971665820979 = 31 \times 41 \times 59^2 \times 653 \times 5897934$

Mr. Bell submitted a computer program for this solution. Copies may be obtained from the editor.

Also solved by L.J. Upton, Steve Feldman, Harry Zaremba, Matthew Fountain, Roger Milkman, Richard Hess, Winslow Hartford, Jim Landau, M.J. Ralph, and the proposer, Walter Nissen.

A/S 2. Find the smallest positive number whose prime factorization consists of the nine digits 1 through 9 and then find the smallest positive number whose prime factorization consists of the ten digits 0 through 9.

I am reprinting Jim Landau's solution, which presents a heurestic approach to attacking the problem. No solver supplied a *proof* of minimality.

To minimize the product of a set of primes consisting of the digits 1 through 9:
(a) 2 should appear as a single-digit prime. If 2 appears in the middle of a prime it takes up an entire decimal place, making the prime approximately 10 times as large as it could be.
(b) 5 should appear as a single-digit prime.
(c) Consider having 3 and 7 as single-digit primes. If 3 or 7 are needed as the last digit of multi-digit primes, try to keep 3 as a single-digit prime.
(d) 9 must be used to terminate a multi-digit prime.
Following (a), (b), (d) and the first sentence of (c), we find
$3,122,490 = 2 \times 3 \times 5 \times 7 \times 14869$
Can we do better? Yes, by going to the 2nd sentence of (c) we find
$2,992,890 = 2 \times 3 \times 5 \times 67 \times 1489$
This is the smallest product I could find. By similar reasoning, I find
$15,618,090 = 2 \times 3 \times 5 \times 487 \times 1069$
for the case with digits 0 through 9.

Also solved by Avi Ornstein, Winslow Hartford, Norman Spencer, M.J. Ralph, Harry Zaremba, Steve Feldman, Richard Hess, Michael Gennert, Matthew Fountain and Nob Yoshigahara.

A/S 3. Divide a two-inch cube into 8 slices with 7 equally spaced cuts. Find the ratio of exposed surface area to number of cuts. Now make 7 cuts in another plane, resulting in 64 little sticks. Again, find the ratio of total area to total number of cuts. Again make 7 cuts, resulting in 512 tiny cubes. The ratios all have similar repeating decimals. Do you dare make another series of cuts into the fourth dimension?

The following solution is from Richard Hess:

$s^1 = 8[4(1/4)2 + 2 \times 2^2]$
$R_1 = S_1/7 = 11.428571428$
$S_2 = 64[4(1/4)2 + 2(1/4)^2]$
$R_2 = S_2/14 = 9.714285714 \ldots$
$S_3 = 512(1/4)^2 \times 6 = 192$
$R_3 = 48/21 = 9.142857142 \ldots$

The editor believes that the suggestion of a fourth dimension was made with tongue in cheek. Several readers, however, carried out the calculation and found that there is no similar repeating decimal.

Also solved by Laurence Bell, Matthew Fountain, Winslow Hartford, Michael Gennert, Jim Landau, Yale Zussman, and the proposer, Phelps Meaker.

A/S 4. Given two evenly matched teams participating in a best-four-out-of-seven series, what is the probability that the team that wins the fifth game of the series will win the series?

Alan Ungar, clearly a baseball fan as well as a master of probability and statistics, submitted a fine solution that goes well beyond the stated problem: Let A represent the event in which the winner of the *fifth* game wins the championship, and B the event in which there is a fifth game. Then by the definition of conditional probability,

$P(A:B) = P(A \times B)/P(B) = P(A)/P(B)$,

where the last equation holds because A is a subset of B. The probability that there is a fifth game, P(B), is equal to the probability that neither team wins the first four games, or

$P(B) = 1 - 2(1/2)^4 = 7/8$.

The event A can occur in three mutually exclusive and exhaustive ways: the winner of the fifth game can win the championship in 5, 6, or 7 games (call these events A1, A2, and A3, respectively). Thus

$P(A) = P(A1) + P(A2) + P(A3)$.

Now, A1 occurs if the fifth game winner wins 3 out of the first 4 games; A2 occurs if it wins 2 out of the first 4, plus the sixth game; and A3 occurs if it wins 2 out of the first 4, loses the sixth, and wins the seventh, or if it wins 1 out of the first 4 and wins the sixth and the seventh. Since it is assumed that the teams are evenly matched, application of the binomial distribution with $p = 1/2$ yields the following values:

$P(A1) = 1/4$, $P(A2) = 3/16$, $P(A3) = 5/32$,

with their sum being 19/32. Thus, we have the result $P(A:B) = P(A)/P(B) = (19/32)/(7/8) = 19/28 = .679$. The empirical probability was 2/3, based on observation of 15 cases. The standard goodness-of-fit test using the chi square distribution with one degree of freedom does not reject the hypothesis that the observed frequency of .667 is significantly different from the theoretical value of .679, at a 95 percent level of confidence. In other words, it is not at all surprising that 10 out of the last 15 fifth-game winners won the championship. It should be noted that a similar analysis shows that the winner of the first, second, third, or fourth games of the series has probability of 21/32 of winning the championship, while the winner of the sixth game has probability of 3/4 of winning the championship. I analyzed the outcome of all the best-of-seven baseball World Series from 1905 through 1982 to see how closely these theoretical probabilities fit the data (Ungar's table is available on request to the *Review* in Cambridge). Although there seems to be a reasonably close match between the theoretical and observed probabilities, there are statistically significant differences which suggest that the assumptions of evenly matched teams and/or of independent Bernoulli trials do not hold. There are, for example, more four-game and seven-game series than one would expect. One possible explanation could be a psychological effect which, for example, saps the morale of teams that go down three games to none, and increases the probability that they lose the deciding fourth game. Indeed, 13 out of the 16 series in which a team lost the first three games ended in four games.

Also solved by Laurence Bell, Winslow Hartford, Jim Landau, Richard Hess, Matthew Fountain, Harry Zaremba, Steve Feldman, Ken Rosato, Peter Tzahnetos, Thomas Jabine, Maria-Elena Cheesman, Kelly Woods, Naomi Markovitz, Judy Badner, Chip Whiting, and the proposer, David De Leeuw.

A/S 5. A "right tetrahedron" has a vertex with three right angles. If a right tetrahedron, A,B,C denote the areas of the three faces that share the "right vertex" and D denotes the area of the face opposite the right vertex, show that $A^2 + B^2 + C^2 = D^2$ (a "Pythagorean Theorem" for areas in 3-space).

Let the length of the leg opposite face A be designated by a, the leg opposite the face B be designated by b, and the length of the leg opposite face C be designated by c. Hence the area of face A is (bc)/2, that of face B is (ac)/2, and that of face C is (ab)/2. Then:

$$A^2 + B^2 + C^2 = (1/4)(a^2b^2 + a^2c^2 + b^2c^2). \quad (1)$$

Let the length of the base of face A be designated by d, the base of face B be designated by e, and the length of the base of face C be designated by f. Area D, the face opposite the right vertex, is given by the following formula:

$$D = (1/4)\sqrt{(d + e + f)(-d + e + f)(d - e + f)(d + e - f)}. \quad (2)$$

Squaring both side of (2) and multiplying the parenthetical expressions yields:

$$D^2 = -d^4 - e^4 - f^4 + 2d^2e^2 + 2d^2f^2 + 2e^2f^2. \quad (3)$$

Since the vertices for faces A, B, and C are all right angles, it follows that

$$d^2 = b^2 + c^2;\ e^2 = a^2 + c^2;\ f^2 = a^2 + b^2. \quad (4)$$

Substituting (4) into (3), obtain

$$D^2 = (1/4)(a^2b^2 + a^2c^2 + b^2c^2). \quad (5)$$

Comparing equations (1) and (5)

$A^2 + B^2 + C^2 = D^2$, as postulated.

Also solved by Naomi Markovitz, Harry Zaremba, Darrell Schmidt, Jerry Grossman and MACSYMA (a computer algebra system—ed.), Matthew Fountain, Ken Rosato, Richard Hess, Winslow Hartford, Jim Landau, Phelps Meaker, Howard Stern, M.J. Ralph, Mary Lindenberg, Jon Peltier, Ronald Goldman, and the proposer, Dennis White.

Better Late Than Never

1980 M/A 3. Harry Kamach and T.R. Keane have submitted a lengthy solution, copies of which may be obtained from the editor.

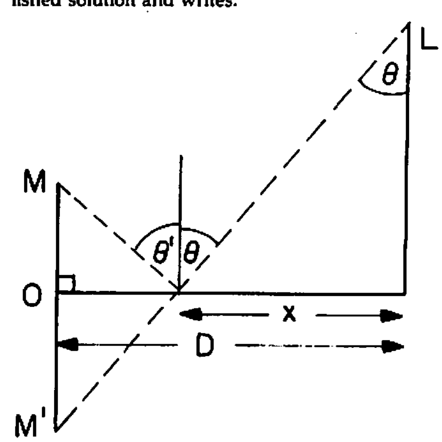
1985 N/D 1. Jim Landau believes that the only thing wrong with his "human friendly" solution is that the penultimate line should read ". . . the two line segments *from* the second set . . ."

1986 APR 1. Joseph Kesselman has responded and included an algorithm, which he attributes to Marc Donner, for calculating the phases of the moon.

Harry Zaremba, the proposer of the problem, noticed the following typographical error in the published solution. The letter S in the two string functions on line 20 should be replaced by the conventional dollar symbol. This applies also to lines 30, 60, 270, and 340 where the functions appear. In lines 40, 50, 70, 80, and 90, the commas should be shifted to the right so that they are located between the quotation marks. In line 100, the letter Y followed by a comma should be added to the right of the first semi-colon. The semi-colon between quotation marks in line 270 should be replaced by a comma. In line 410, the "greater than" symbol should be replaced by a "less than" symbol.

APR 4. Peter Silverberg has responded.

M/J 4. Martin Deutsch is not pleased with the published solution and writes:



The *image* of M is located at M' and does not move at all (unless the floor is suddenly tilted!). The point K is the intersection of the line of sight L-M' with the floor. We "note from physics" that $\theta = \theta'$. Physics does *not* say anything about the relationship between θ, x, and D. That is high school geometry, as is the following statement: Two triangles having the same values for two of their angles are similar triangles. Hence:

$x/(D - x) = 120/62$,

from which it follows that

$x/D = 120/182$.

This ratio holds for all values of D (even negative ones). Hence the change in x and D for any time interval is also in this ratio:

$\Delta x/\Delta D = 120/182$, and therefore

$R_x/R_D = 120/182 = 0.659$, and

$R_x = 0.695(42) = 24.692$

As a professor of physics at M.I.T., I may hold a distorted view of the mind of a typical reader of *Technology Review*. Still, I don't see how the introduction of irrelevant square roots and trigonometric functions makes the answer easier to grasp by anybody.

JUL 3. David Simen has responded.

JUL SD 2. Robert Parry notes that the midpoint is actually *south* in the northern hemisphere.

A/S SD2. Kelly Woods notes that the sequence for part (a) can be extended to the left to begin 5, 10, 11, . . .

Proposers' Solutions to Speed Problems

SD 1. The largest number is 7^{691}, which is approximately 10^{1082}. The smallest non-negative number is $0 = (9 - 8 - 1) \times 7$.

SD 2. From 1 May 1988 through 30 June.