ALLAN J. GOTTLIEB

A Tight Spot for the Piano Movers

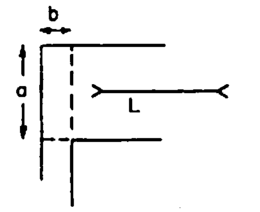
Several readers have made comments that I wish to pass on. Longtime contributor Frank Rubin runs a series of contest puzzles. For details write to "The Contest Center," P.O. Box 1660, Wappingers Falls, N.Y. 12590. Mary Lindenberg's contributions to *Puzzle Corner* have been cited by her local newspaper. Finally, Dudley Church seems to be starting a fan club; he writes, "If Harry Zaremba is not a mythical character, he is undoubtedly one of the premier puzzle solvers of the age."

Problems

N/D 1. Lawrence Kells writes that a friend of his directed a duplicate bridge tournament in which every hand was played 20 times. Not being noted for its bidding expertise, this section produced a strange result on Board 14. At four of the tables the final contract was one club, played once from each of the four sides. At four of the other tables it was played at one diamond once from each side. Likewise, at the remaining tables it was played once from each side at one heart, one spade, and one no-trump. Every one of these contracts was set. Analysis of the hand proved that none of the declarers made a mistake in play. Unfortunately, Mr. Kells failed to see what the deal was. Can you reconstruct it? (That is, a deal where any contract anybody bids can be set no matter how hard succesful bidder tries to make it.)

N/D 2. Our next problem is from Harry Zaremba. A certain polyhedron has nine vertices, and each of its faces is a triangle. How many faces does the figure have? If six faces meet at each of three vertices, what common number of faces meet at each of the other vertices?

N/D 3. Ruben Cohen offers a problem he hopes will be helpful to students who need to change apartments each year. Since this is one form of the "piano-movers" problem, the students are presumably music majors. Mr. Cohen writes that some students need to move a board of length L through a corner as shown below.



What is the maximum length board that can pass through?

N/D 4. 1985 Oct 4 reminds Albert Mullin of the following problem, first submitted to *American Mathematical Monthly* by Fred Jamison in 1951:

Any one of a group of airplanes may be refueled from any other. Each has a fuel capacity sufficient for a flight one-fifth the distance around the earth. Assuming that all have the same constant ground speed and the same rate of fuel consumption, that the only landing place and the only available fuel supply are at the home base, and that refueling time is negligible, find the minimum number of planes necessary so that one plane may fly around the earth and all return home safely.

N/D 5. A former colleague of mine, Ned Staples, would like you to find a method of converting an arbitrary (legal) position in a tower of Hanoi puzzle into another arbitrary position. In the tower of Hanoi puzzle, we have M discs of differing radii distributed on three pegs with no disc on top of a smaller one.

Speed Department

SD 1. Phelps Meaker recalls that the formula $\cos^2\pi/N$ came up when studying regular polygons having N sides. In what way?

SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, THE COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y. 10012.

SD 2. L. Upton wants an economical representation for 2.34567 (to the nearest fifth place) and recommends using a hand calculator.

Solutions

JUL 1. The contract is six hearts by South, East having bid spades. With an opening lead of ♠7, how can South fulfill the contract?

	♠ A J 9 8 ♥ Q 9 ♦ Q 5 4 2 ♣ 4 3 2	
♠ 7 6 5 ♥ 3 2 ♦ J 10 9 6 ♣ J 9 6 5		♠ K Q 10 4 3 ♥ 8 7 6 ♦ A 3 ♣ Q 10 8
	♠ 2 ♥ A K J 10 5 4 ♦ K 8 7 ♣ A K 7	

The following solution is from Jacob Bergmann: South takes the opening trick with the ♠A and leads a low diamond. If East does not win immediately, you win the ♦K; continue with a low diamond from both hands and you lose that trick. In either case, you establish two top diamond winners and retain the ♦Q in dummy. Now let East return a club (his best shot). Win the ♣A and run off all the trumps, reducing to

♠ J
♥ -
♦ Q 5
♣ 4

♠ -
♥ 5
♦ 8
♣ K 7

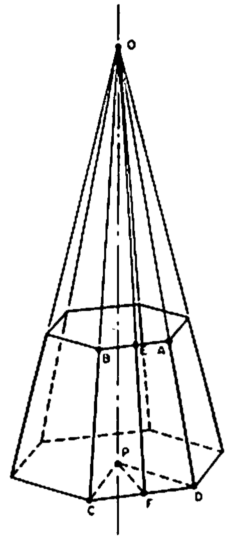
The type B1 double squeeze is now in order. On the last trump, West must reduce to a singleton club in order to guard diamonds; you discard the ♦5 from dummy, cross to the ♦Q, and East is squeezed in the black suits.

Also solved by Ned Staples, Joe Feil, Bruce Levy, Douglas VanPatter, Matthew Fountain, Richard Hess, Jacob Bergmann, Robert Bart, Timothy Maloney, Yale Zussman, Michael Strieby, Warren Himmelberger, and the proposer, Winslow Hartford.

JUL 2. A six-sided lampshade assembled from parchment trapezoids is larger at the bottom than at the top. The two parallel sides and the height of the trapezoids are in the ratio 2:3:4. How much does a side slope from the vertical?

The following response is from James Abbott: As usual, the solution took me a lot less time than the documentation! For this one, I made a set of panels according to the specification and put them together with tape. Some of the resulting shapes are fascinating possibilities for lampshades. The structure as specified is not rigid and can assume any shape with a six-sided cross section (including re-entrant shapes such as a three-pointed star). However, for lack of any other specification I am assuming it to be a frustum of a regular hexagonal pyramid.

The figure is oriented to show the center of the base and associated construction lines useful to the solution. In the figure, ABCD is one of the six trapezoidal faces, O the vertex of the pyramid, and P the center of the base. Since the base is by definition a regular hexagon, it follows that PCD is an equilateral triangle. Let E and F be the respective midpoints of AB and CD. Then, since in a regular pyramid all faces are congruent by definition, OC = OD. Also, PC = PD since PCD is equilateral. Triangles OFC and OFD are congruent; hence angles OFC and OFD are equal right angles and OF is perpendicular to CD. Similarly, PF is also perpendicular to CD. Therefore, the angle PFO measures the dihedral angle between the planes determined by OCD and PCD. The complement of this angle, or POF, will be the slope asked for.



The statement of the problem gives the ratio of AB:CD:EF as 2:3:4. For simplicity of expression let us specify a unit such as that AB = 2, CD = 3, and EF = 4. Then by the rules of similar figures, OE/OF = 2/3, or OE = (2OF)/3.
Since EF = OF − OE (from the figure), we have EF = OF − (2OF)/3 = OF/3, whence
OF = 3EF = 3 × 4 = 12.
Since PCD is an equilateral triangle, angle PCF = 60°.
PF = PC sin 60° = CD sin 60° = 2.5981.
Finally, angle POF = arcsin (2.5981/12) = 12.5039°.

Also solved by Ned Staples, Jacob Bergmann, Greg Spradlin, Harry Zaremba, James Abbott, John Cushnie, Ken Rosato, Kurt Eichenberger, Mary Lindenberg, Matthew Fountain, Peter Wender, Richard Hess, Robert Bart, Ronald Martin, Michael Gennert, Steve Feldman, Winslow Hartford, Naomi Markovitz, P. Michael Jung, John Langhaar, Dennis White, Yale Zussman, and the proposer, Phelps Meaker.

JUL 3. Find a pattern or better yet a closed form for f(n) defined by:
f(1) = 1
f(n) = f[n − f(n − 1)] + 1 for n > 1

George Thomas sent us the following solution. The pattern is 1,2,2,3,3,3,4,4,4,4,5,5,5,5,5,..., i.e. one 1, then two 2's, etc., and a closed form that fits that pattern is

$$f(n) = \left[\frac{\sqrt{8n+1}-1}{2}\right]$$

where [x] means the greatest integer in x. After discovering the pattern (and proving it by mathematical induction), it seemed a good idea to look at the triangular numbers
$x_1 = 1$, $x_2 = 1 + 2 = 3$, $x_3 = 1 + 2 + 3 = 6$, etc.
$x_m = [m(m + 1)]/2$
because the rule for defining f(n) fits the same pattern as g(n) = m when n satisfies the inequalities
$[m(m - 1)]/2 < n \leq [m(m + 1)]/2$.
With a bit of algebra, these inequalities define m in terms of n as follows:
$(x - 1)/2 \leq m < (x + 1)/2$, where $x = \sqrt{8n + 1}$.
A bit of fooling around with greatest integers led to the final formula for f(n).

Alan Prince supplied the inductive proof referred to above. He notes first that f(2) = 2, so that the sequence does start off with just one 1. Suppose that the suggested pattern holds up to some N − 1; we show that it holds also for N. Suppose that f(N − 1) is the rth k in the sequence of k's, where r ≤ k. Notice that by hypothesis the k-run begins with f(N − r). Now, N − f(N − 1) = N − k. If r < k, f(N − k) = k − 1, and f(N) = k. If r = k, then f(N − k) = f(N − r) = k and f(N) = k + 1. Thus, the k-run will have exactly k members, for all k, and is succeeded by the (k + 1)-run.

Also solved by Jacob Bergmann, Alan Unger, Avi Ornstein, Bruce Levy, Charles Sutton, Frank Carbin, George Thomas, Greg Huber, Greg Spradlin, Harry Zaremba, Matthew Fountain, Oren Cheyette, Ray Kinsley, Richard Hess, Robert Bart, Dennis White, Sidney Darlington, Michael Gennert, John Langhaar, Winslow Hartford, P. Michael Jung, Steve Feldman, Alan Prince, Timothy Maloney, Naomi Markovitz, Yale Zussman, Stanley Liu, and Stephan Goldstein.

JUL 4. A spy received a rain-soaked letter. The stamp was gone, and all the postmark was obliterated except two continuous letters, "ON." The contents bore no letterhead, but on decoding the letter the spy concluded it must have been mailed in Bonn, London, or Washington. What is the probability it was mailed in London?

There seem to be two interpretations possible. If one assumes that a certain percentage of the envelope was legible, then London, with two ON's, is twice as likely as either Bonn or Washington to have an ON in the legible portion. In this case the probability that the letter came from London is (by Bayes Theorem)
$2x/(2x + x + x) = .5$
If, however, one assumes that the rain was a process that managed to leave a random two consecutive letters legible, we must note that the number of possible consecutive pairs in a word is one less than the number of letters in the word. So for London, the chances of the pair being ON are 2/5, and the probability of the letter coming from London is
$2/5 \div (2/5 + 1/3 + 1/9) = 9/19$.
Although this second interpretation gives a better problem, the first seems to match the behavior of rain more closely.

Also solved by Jacob Bergmann, Chip Whiting, Alan Marzilli, Bruce Levy, Frank Carbin, Greg Huber, Greg Spradlin, P. Michael Jung, Harry Zaremba, Ken Rosato, Mary Lindenber Matthew Fountain, Richard Hess, John Langhaar, Peter Wender, Robert Bart, Dennis White, Sidney Shapiro, Winslow Hartford, and Yale Zussman.

JUL 5. I remove one M&M at a time at random from a bag and place it on the table. Whenever there are two of the same color on the table, I eat those two. After I have removed 100 M&M's from the bag, how many might I expect to find left on the table? (Assume there are five colors altogether, distributed equally.) After 101? If brown ones are twice as common as the other colors, how do the expected values change after 100 and 101?

Oren Cheyette, assuming that 100 is large enough to ignore the initial state, gives an easy proof that the answer is 2.4 in all cases. Alan Unger gives us the exact solution. Mr. Cheyette writes:
The following solution assumes an infinite bag of M&Ms (so that successive draws are independent). The expectation value of the number of M&Ms on the table after a large number of draws (whether odd or even) is 5/2 (1/2 for each color) regardless of the relative frequency of each color. The probabilities for a particular number of M&Ms being on the table are, after an even number of draws,
$P_0 = 1/16$, $P_2 = 10/16$, $P_4 = 5/16$, all others 0.
and after an odd number of draws
$P_1 = 5/16$, $P_3 = 10/16$, $P_5 = 1/16$, all others 0.
After a large number of draws, the M&Ms on the table have "forgotten" the initial conditions (that there were originally none on the table), and then half the time there is one M&M of any given color, half the time there is none. The probability distributions given above may be obtained by looking at the odds for various numbers of heads among five flipped coins, subject to the "parity rule" that there be an even number of M&Ms after an even number of draws, odd after an odd number of draws.

Mr. Unger writes:
Assume there are an infinite number of red, orange, yellow, green, and brown M&Ms in the bag. In the first part of the problem the probability of selecting each color is the same; p = 1/5. Let X(i,j) be a random variable representing the number of M&Ms of color i left on the table after j draws from the bag. By the conditions of the problem there can be either 0 or 1 of each color on the table at any time, and the probabilities of each event are:

Probability that an even number of M&Ms of color i have been selected after j random draws:
P(X(i,j) = 0) = Pij
Probability that an odd number of M&Ms of color i have been selected after k random draws:
P(X(i,j) = 1) = Qij
The expected total number of M&Ms remaining on the table after 100 random draws (call this N(100)) is EN(100)
= E(X(1,100) + X(2,100) + X(3,100) + X(4,100) + X(5,100)
= EXC(1,100) + EX(2,100) + EX(3,100) + EX(4,100) + EX(5,100)
= 5EX(1,100)
since the X(i,100) are identically distributed. Now the expected value of X(1,100) is by definition
EX(1,100) = O P(X(1,000) = 0) + 1 P(X(1,100) = 1) = O + P(X(1,100) = 1) = Q1j = Qj
Thus, EN(100) = 5 Qj.
The expected number of M&Ms remaining on the table after 100 draws is equal to five times the probability that a particular color has been selected an odd number of times. To compute this probability consider the following recursion relation:
Qj + 1 = (1 − Qj)p + Qj(1 − p).
This is derived by noting that there are two mutually exclusive events that lead to the selection of a particular color an odd number of times after j + 1 draws: the color could have been selected an even number of times after j draws, followed by its selection on draw j + 1; or it could have been selected an odd number of times after j draws, followed by its non-selection on draw j + 1. Adding the probabilities of these events yields the above equation. It is easily solved by using the method of probability generating functions: multiply both sides by x^j, sum from j = 0 to ∞, solve for $g(x) = \Sigma Q_j x^j$, and equate the coefficient of x^j to Qj. The result is
$Qj = [1 - (1 - 2p)^j]/2$
Thus the expected number of M&Ms remaining on the table after 100 draws is
$EN(100) = 5Q100 = 2.5 - 5(.6)^{100}/2 \approx 2.5 - 1.6 \times 10^{-22}$.
and after 101 draws it is
$EN(101) = 2.5 - 5(.6)^{101}/2 \approx 2.5 - 9.8 \times 10^{-23}$.
In the second half of the problem it is assumed that the probability of selecting a brown M&M is twice as great as the probability for each of the other colors. This implies that the selection probability for browns is 1/3, compared to 1/6 for the other four colors. The equation for the expected value of the number of M&Ms remaining on the table after j draws becomes
EN(j) = Qj(for brown) + 4Qj(others).
Substituting for Qj yields
$EN(j) = [1 - (1/3)^j]/2 + 4[1 - (2/3)^j]/2 = 2.5 - (1/3)^j/2 - 4(2/3)^j/2$.
Thus after 100 draws the expected number of M&Ms on the table is
$EN(100) \approx 2.5 - 9.8 \times 10^{-18}$,
and after 101 draws it is
$EN(101) \approx 2.5 - 6.6 \times 10^{-18}$.
Note that the probability is very close to 1/2 that a particular color will be on the table after 100 or 101 draws, whether each color is equiprobable as in the first part of the problem or brown is twice as likely as in the second part. Using the approximation Qj = 1/2, it is easy to derive the distribution of M&Ms on the table after n draws. Given n even, the conditional probability of any combination of colors on the table is 1/16. Since there are five ways to select four colors from five, ten ways to select two colors from five, and one way to select zero colors from five, the probability distribution of M&Ms on the table after 100 draws is as follows:
P[N(100) = 0] = 1/16
P[N(100) = 2] = 5/8
P[N(100) = 4] = 5/16.
By a similar argument the probability distribution after 101 draws is
P(N[101) = 1] = 5/16
P(N[101) = 3] = 5/8
P(N[101) = 5] = 1/16

Also solved by Jacob Bergmann, Henry Lieberman, Ronald Goldman, Chip Whiting, Alan Unger, Frank Carbin, Greg Spradlin, Harry Zaremba, Matthew Fountain, John Langhaar, Oren Cheyette, Dennis White, Peter Kramer, Raymond Kinsley, Richard Hess, Yale Zussman, Robert Bart, Sidney Shapiro, Winslow Hartford, and the proposer, Allen Wiegner.

Better Late Than Never

85 OCT 3. Dennis White has solved several special cases; the general problem remains open.

1986 F/M 4. Stephan Goldstein and Sidney Hendrickson are not happy with the published solution. They also disagree with each other. Mr. Goldstein believes that the original potential energy was .25 mg. Since the final kinetic energy is $.5mv^2$, he concludes that $v^2 = g/2$ or v is approximately 22.14 cm/sec. Mr. Hendrickson writes:

The solution given has two errors. The first is the statement that the wall imparts no energy to the stick. The force the wall exerts on the stick is the horizontal force which causes it to move to the right. F x dx is obviously non-zero. Fortunately, this fact was not used in further analysis of the problem. The basic concept used in this analysis, conservation of energy, is certainly appropriate. The execution was in error, however. Energy lost *is* indeed potential energy and is given by $mg(y_o - Y)/2$, where Y is the height of the end of the stick on the wall. The potential energy *is* converted into kinetic energy. The energy of translation *is* indeed $mV^2/2$. But kinetic energy *is not* just energy of translation. It also includes rotational energy. This had been totally forgotten. The distance of the bottom end of the stick from the wall is given by X. The positions and velocities of the ends are, in (x,y) coordinates, upper end: (0,Y), $(0, -V_y)$, lower end: (X,0), $(V_x,0)$, and the position and velocity of the center-of-mass (COM) are (X/2,Y/2), $(V_x/2, -V_y/2)$. The magnitude of the velocity of the COM is $V = \sqrt{V_x^2 + V_y^2}/2$. With respect to the COM, the stick ends are moving with velocity, upper: $(-V_x/2, V_y/2)$, lower: $(V_x/2, V_y/2)$. A little arithmetic will show that the direction of movement of the ends is perpendicular to the stick as expected, since the stick is rigid and the ends would not move toward (contract) or away from (expand) the center. Some arithmetic will also show that the magnitude of the velocity of the ends with respect to the COM is also V! Hence the angular velocity $w = V/(L/2) = 2V/L$. The energy of rotation is $E_r = w^2I/2 = (4V^2/L^2)(mL^2/12)/2 = mV^2/6$, where I is the moment of inertia of a rod of length L, spinning about its center, and of uniform density with mass m.

As with the previous analysis, lost potential energy is equated with kinetic energy gained: $Mg(Y_o - Y)/2 = E_t + E_r = 2mV^2/3$, but this time the kinetic energy contains the rotational energy term. As with the previous analysis, $2V^2 = (V_x/Y)^2$; hence $V_x = \sqrt{3/4}\, Y\sqrt{g(Y_o - Y)}$. This is $\sqrt{3/4}$ of the final expression found previously. A similar analysis to previous $dV_x/dt = 0$ when $Y = 2Y_o/3$, exactly at the same point as before! (My Y_o is the original height of the upper end of the stick, 50 cm, so Y = 100/3.) The difference is that V_x is $\sqrt{3/4}$ of the value found previously, or approximately, 32.0 cm/sec.

JUL SD2. Although I normally do not print comments on speed problems, I could not resist the following remark from Charlie Bostick:
I usually ask this question about analog watches and then, when the solution is exposed, ask if there is a corresponding algorithm for digital watches. Yes, there is! One notes the time on the digital watch and mentally pictures the hands as they would appear on the analog watch. Then the same rule is followed. If digital watches ever completely replace analog watches, then people may never be able to picture the hands mentally. At that time in the future, unless compasses are used, people will have the time but not the inclination.

Proposer's Solutions to Speed Problems

SD 1. The ratio of the areas of the inscribed to circumscribed circles.

SD 2. $2^{1^{23}}$.