ALLAN J. GOTTLIEB

Happy New Year: 1 to 100 in 1, 9, 8, and 6!

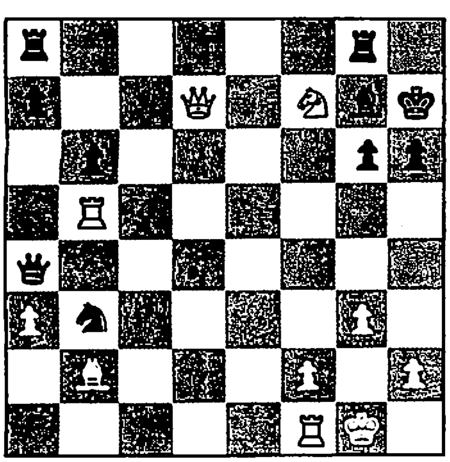
I have some very good news to report. At 1:06 a.m. on September 24, 1985, our second child was born. As with our first child David, my wife Alice and I attended Lamaze classes (this time just a "refresher" course) and so once again had a good "birthing experience." To paraphrase my own words from three and a half years ago, this column is dedicated to the fruits of Alice's labor, Michael Bendix Gottlieb.

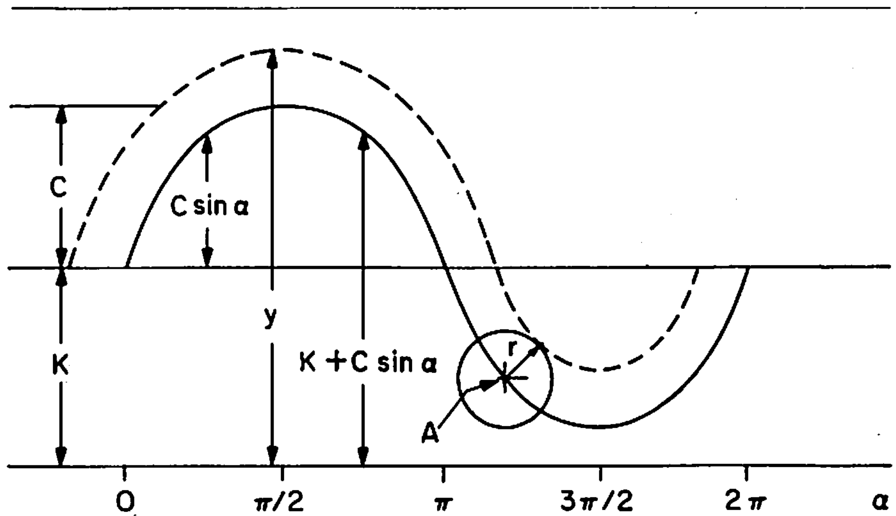
This being the first issue of another year, we again offer a "yearly problem" in which you are to express small integers in terms of the digits of the new year (1, 9, 8, and 6).

Problems.

Y1986. Form as many as possible of the integers from 1 to 100 using the digits 1, 9, 8, and 6 exactly once each and the operators +, −, × (multiplication), / (division), and exponentiation. We desire solutions containing the minimum number of operators; and, among solutions having a given number of operators, those using the digits in the order 1, 9, 8, and 6 are preferred. Parentheses may be used for grouping; they do not count as operators.

JAN 1. We begin with a ("no gimmicks") chess problem from Craig Presson, who requires White to move and mate in four.

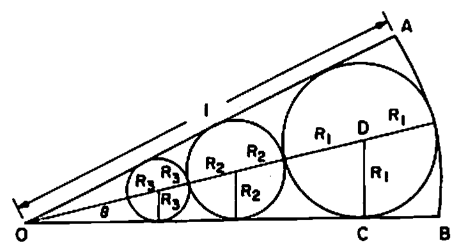




JAN 2. Philip Hogin offers $100 for a solution to
$y = f(\alpha)$
for given values of K, C, and r in the diagram at the top of this column. This is required for the design of a cylindrical cam which produces simple harmonic motion of the center of the follower (point "A"). The expression $y = f(\alpha)$ will be used to program a laser cutting machine. In the prototype design, the diameter of the follower is .5" (r = .250"), C = .75", K = 1.25", and the cam circumference is 3". If an exact solution is complicated, an accurate approximation would be acceptable.

JAN 3. George Byrd recalls, from the Dover *Diversions and Digression*, that the six-digit number 142857 has an interesting property. Consider
One times the number is 142857.
Two times the number is 285714.
Three times the number is 428571.
Four times the number is 571428.
Five times the number is 714285.
Six times the number is 851428.
Note that all these numbers are rotated versions of the original. Mr. Byrd would like to know if there are other numbers, X, having the property that multiplication by any positive integer not exceeding the number of digits of X produces a rotated version of X.

JAN 4. Howard Stern poses an interesting area maximization problem:



Consider the sector OAB of a circle of unit radius with angle 2θ. We can "fill up" the sector with an infinite series of circles with radii R_1, R_2. . . . Obviously, the ratio of the sum of the circles' areas to the sector's area is less than 1. For what θ is this ratio the largest and what is its value?

Speed Department

SD 1. Our first speed problem was attributed to Donald Knuth in the October

SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, THE COURANT INSTITUTE, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y. 10012.

1985 *American Mathematical Monthly*. What is the next number in the sequence that begins with F4E, S9, SE5EN,...?

SD 2. We close with a question from the "freshman quiz" given by *The Tech*, M.I.T.'s student newspaper. I believe this problem actually originated in *Puzzle Corner* about ten or fifteen years ago. Translate the following into a limerick:

$$(12 + 144 + 20 + 3\sqrt{4})/7 + 5 \times 11 = 9^2 + 0$$

Solutions

Y1985. This is the same problem as Y1986 (see above) with only one digit changed.

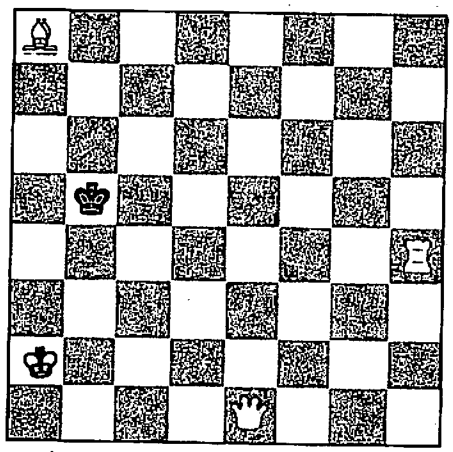
The following solution is from Avi Ornstein, who indicates with bold face those solutions using the digits in order:

1 **1^{985}**
2 $1 + (9 - 8)^5$
3 $5 - 18/9$
4 $81/9 - 5$
5 **$1^{98} \times 5$**
6 $91 - 85$
7 $18/9 + 5$
8 $9 - 1^{58}$
9 $1^{58} \times 9$
10 $18/9 \times 5$
11 **$-1 + 9 + 8 - 5$**
12 $(95 + 1)/8$
13 **$1 + 9 + 8 - 5$**
14 $95 - 81$
15 $15 \times (9 - 8)$
16 $19 - 8 + 5$
17 $8 + 9 \times 1^5$
18 $(89 + 1)/5$
19
20
21 $9 \times 8 - 51$ or $8 \times 5 - 19$
22 $81 - 59$
23 **$1 + 9 + 8 + 5$**
24 $8 \times (9 - 1 - 5)$
25 $5^{(18/9)}$
26 $9 \times (8 - 5) - 1$
27 $9 \times 5 - 18$
28 $(8 - 1) \times (9 - 5)$
29
30 **$(1 + 9) \times (8 - 5)$**
31 $8 \times 5 - 1 \times 9$
32 $19 + 8 + 5$
33 $91 - 58$
34 $51 - 9 - 8$
35 $(8 - 1^9) \times 5$
36 $81 - 9 \times 5$
37 $9 \times 5 - 8 \times 1$
38 $89 - 51$
39 **$-1^9 + 8 \times 5$**
40 **$1^9 \times 8 \times 5$**
41 **$1^9 + 8 \times 5$**
42 $(1 + 9) \times 5 - 8$
43
44 $(8 - 1) \times 5 + 9$
45 $81/9 \times 5$
46 $5 \times 9 + 1^8$
47 $98 - 51$
48 $58 - 9 - 1$
49 $58 - 9 \times 1$
50 $59 - 8 - 1$
51 $59 - 8 \times 1$
52 $59 - 8 + 1$
53 **$9 \times 5 + 8 \times 1$**
54 **$9 \times 5 + 8 + 1$**
55 **$(19 - 8) \times 5$**
56
57 **$19 \times (8 - 5)$**
58 $59 - 1^8$
59 **$19 + 8 \times 5$**
60 $59 + 1^8$
61
62 $(5 + 1) \times 9 + 8$
63 $9 \times 5 + 18$
64 $8 \times (9 - 1^5)$
65
66 $85 - 19$
67 $59 + 8 \times 1$
68 $59 + 8 + 1$
69 $8 \times (9 - 1) + 5$
70
71 $8 \times 9 - 1^5$
72 $18 \times (9 - 5)$
73 $8 \times 9 + 1^5$
74 $89 - 15$
75 $85 - 1 - 9$
76 $85 - 9 \times 1$
77 $19 + 58$
78 $91 - 8 - 5$
79
80 $(1^5 + 9) \times 8$
81 $5 \times 18 - 9$
82
83 $98 - 15$
84 $89 - 5 \times 1$
85 $89 - 5 + 1$
86 $95 - 8 - 1$
87 $95 - 8 \times 1$
88 $95 - 8 + 1$
89 89×1^5
90 $89 + 1^5$
91
92 $98 - 5 - 1$
93 **$1^{98} - 5$**
94 **$1 + 98 - 5$** or **$1 \times 9 + 85$**
95 **$1 + 9 + 85$**
96 $95 + 1^8$
97 $98 - 1^5$
98 98×1^5
99 $98 + 1^5$
100

Also solved by Marion Berger, Jim Landau, Steve Feldman, Robert Kruger, Alan Kätzenstein, Harry Zaremba, Allen Tracht, A. Holt, Frederich Furland, Dudley Church, Randall Whitman, Rik Anderson, Roger Wiethoff, Phelps Meaker, Peter Silverberg, George Aronson, and Ellen Kranzer.

A/S 1. White is to move and force a mate in two.

The following solution is from Gary Schlegelmilch and Marc Campbell:

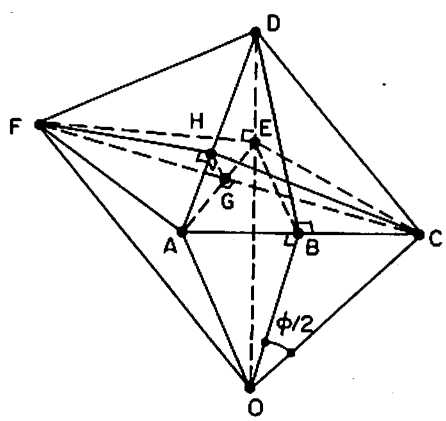


White must move Rook H4-A4.
If Black moves KxR (B5-A4), then White moves Bishop A8-C6 mate.
If Black moves King B5-B6, then White moves Queen E1-A5 mate.
If Black moves King B5-C5, then White moves Queen E1-B4 mate.

Also solved by Richard Hess, Alison Prince, Charles Rivers, Ruben Cohen, Nolan Kagetsu, David Cohen, Ron Raines, Jerry Horton, Jacob Bergmann, Matthew Fountain, Gardner Perry, Thomas Chang, Edward Gaillard, Ronald Ort, and William Maimone.

A/S 2. A reader wanted an approximation of a four-foot sphere for light-integration measurement. He made a dodecahedron of sheet metal, except that instead of flat surfaces, he substituted low five-sided pyramids. How should he have desiged the pyramids so that all the dihedral angles were equal? The following solution is from Harry Zaremba:

All the dihedral angles should equal 156°43′6.8″. Each vertex at which five faces of a pyramid intersect should be 0.359 feet above the base of the pyramid, and each edge intersecting at the vertices should be 1.266 feet long. The foregoing values were determined as follows.



In the figure shown, 0 is the center of the sphere which circumscribes the dodecahedron, D is the vertex of a pyramid whose base is represented partially by triangles FEA and AEC, and the plane of triangle FHC is perpendicular to edge AD of the pyramid. Angles FHG, CHG, and DBO equal β which is one-half of the dihedral angle between the pyramid faces, angle EBO equals θ, and angle DBE equals β − θ. In congruent right triangles ABE and CBE, angles AEB and CEB = 36° and angles EAB and ECB = 54°. The radii FO, AO, and CO of the sphere equal 2 feet. From the (36°, 90°, 60°) spherical right triangle, which is not shown above the planar right triangle EBC, we have

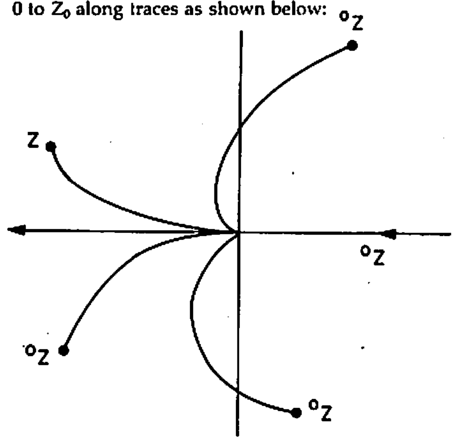
cos36° = cos(φ/2) sin60°, or

$(\phi/2) = 20°54'18.57''$.
From the right triangle CBO,
$BC = AB = 2\sin(\phi/2) = 0.71364$
$BO = 2\cos(\phi/2) = 1.86834$
$AC = 2BC = 1.42729$.
In right triangle EBC,
$EB = BC/\tan 36° = 0.98225$
$EC = EA = (EB^2 + BC^2)^{1/2} = 0.98225$
$EC = EA = (EB^2 + BC^2)^{1/2} = 1.21412$
and in right triangle BEO,
$\cos\theta = EB/BO = 0.52573$, or
$\theta = 58°16'57.1''$
From right triangle BED, the distance of the pyramid's vertex above its pentagonal base is
$DE = EB\tan(\beta - \theta)$. (1)
Now from right triangle AGC,
$GC = AC\sin 54° = 1.15470$.
In right triangle CGH,
$CH = GC/\sin\beta$ and
$GH = GC/\tan\beta$
and from right triangle AHC,
$AH = (AC^2 - CH^2)^{1/2} = [AC^2 - (GC^2)/\sin^2\beta]^{1/2}$
Since right triangles AHG and RED are similar, we have,
$AH/GH = AE/DE$.
After substituting DE, GH, and AH into the equation above and noting that
$AC = GC\ E/EB$, we get
$[AC^2 - (GC^2)/\sin^2\beta]^{1/2} = AC/(\tan\beta\tan(\beta - \theta)$; or
$(1.42729^2 - 1.15470^2/\sin^2\beta)^{1/2} - 1.42729/[\tan\beta\tan(\beta - 58.282526)] = 0$.
Solution to above equation is, $\beta = 78.35928°$.
Thus the dihedral angles are $2\beta = 156°43'6.8''$.
From equation (1) above,
$DE = 0.359$ feet
and the length of the pyramid edges is
$AD = (DE^2 + EA^2)^{1/2} = 1.266$ feet

Also solved by Dennis White, Matthew Fountain, Thomas Chang, and Winslow Hartford.

A/S 3. A four-port device contains only passive linear circuit elements. Using a fixed-frequency sine-wave generator, you can measure the complex impedance Z between two of the terminals, A and B. A variable resistor R is connected across the other two terminals, C and D. What is the locus in the complex plane traced out by Z as R varies from zero (short circuit) to infinity (open circuit)?

The proposer, Randy Barron, claims (thus far without proof) that the curve traced is an arc of a circle. However, Richard Hess has sent us the following non-circular solution:
(1) Assume $Z_0 = x + iy$ and R is in parallel with it. Then, the totalimpedance is
$Z = Z_0R/(R + Z_0)$
$= (Rx + iRy)/(R + x + iy)$
$= (R^2x + Rx^2 + Ry^2 + iyR^2)/(R^2 + ZRx + x^2 + y^2)$
(2) The trace of Z as R goes from 0 to ∞, Z goes from 0 to Z_0 along traces as shown below:



A/S 4. Fill in the boxes with the digits 1,2,3...,9:
□□min:□□sec x □ = □□min: □□sec

William Maimone submitted the following solution:
The temporal multiplication problem may be reduced through the application of the following steps:

[x1] [x2] : [x3] [x4] × [x5] = [x6] [x7] : [x8] [x9]

1. Since the maximum number of minutes or seconds is 60, x1, x3, and x8 must all be ≤ 5.
2. Since no number can be repeated, x5 $\neq 1$.
3. Since x5 > 1, × 1 $\leq$ 2.
4. Since x1 = 2 would force x5 = 3, which would yield a product greater than 60 minutes, x1 must be 1.
5. Since x3, x5, and x8 must all be between 2 and 5, no other positions can be between 2 and 5.
6. Since the smallest x3x4 is 16, and 4 × 16 > 60, x5 < 4. Therefore x5 must be 2 or 3.
7. Since the smallest x5 × the smallest x1x2 = 32, x6 $\neq$ 2.
8. Since no product of 2 or 3 and 7 yields a valid x9, x4 $\neq$ 7. This leaves only 2, 3, 4 or 5 for x3 and 6, 8, or 9 for x4.
9. Since letting x3 = 3 would force x5 = 2, and 2 × 36, 38, or 39 would force x8 = 1, x3 $\neq$ 3.
10. Since letting x3 =2 would force x5 = 3, and x × 26, 28, or 29 would force x8 = 1, x3 $\neq$ 2.
11. Since no number can be used twice, most possible seconds left multiplied by 2 or 3 can be eliminated, leaving valid x3x4 × x5 combinations of 48 × 2, 49 × 2, 49 × 3, and 56 × 3.
12. Obviously, x3 $\neq$ 2, x3 $\neq$ 3, and due to the products of these combinations, x8 $\neq$ 5, and x9 $\neq$ 9.
13. Since x3 and x6 cover 4 and 5, x8 $\neq$ 4.
14. Since x8 $\neq$ 4, 56 × 3 cannot work, so x3 $\neq$ 5, and x4 $\neq$ 6. Then x3 = 4 and x6 = 5.
15. Since x4 is 6, 8, or 9 and x9 is 6, 7, or 8, x5 $\neq$ 2, so x5 = 3 and x8 = 2.
16. Since x6 = 5 and x5 = 3, x2 $\neq$ 6.
17. Finally, x4 = 9, x9 = 7, x2 = 8, x7 =6. So the result is:

[1] [8] : [4] [9] × [3] = [5] [6] : [2] [7]

Also solved by Howard Stern, John T. Coleman, Ronald Ort, Dennis White, Harry Zaremba, David Cohen, Avi Ornstein, Matthew Fountain, Winslow Hartford, Steve Feldman, Rik Anderson, Jacob Bergmann, Charles Rivers, Victor Newton, Norman Spencer, George Peckar, Jerry Horton, Mary Lindenberg, Michael Hennessey, John Cushnie, and Ross Hoffman.

A/S 5. It began to snow on a certain morning, and the snow continued to fall steadily throughout the day. At noon, a snowplow started to clear a road at a constant rate in terms of the volume of snow removed per hour. The snowplow cleared two miles by 2 p.m. and one more mile by 4 p.m. At what time had the snowstorm begun?

This final solution is from John Cushnie:
Assumptions:
t = hours after start of snow
T = hours of snow before noon
D = snowfall rate, feet per hour
C = flow capacity (12" path), cubic feet per hour
Solution:
At t hours, snow depth = Dt feet and plow velocity = C/Dt feet per hour
d(distance) = (velocity)dt = (C/D)(dt/t)

$\text{Distance} = (C/D) \times \int_{t_1}^{t_2}(dt/t) =$
$(C/D)(\ln t)_{t_1}^{t_2}$
For noon to 2 p.m., $t_1 = T$ and $t_2 = T + 2$ and distance =
$(C/D)(\ln t)_T^{T+2}$
For 2 p.m. to 4 p.m., $t_1 = T + 2$ and $t_2 = T + 4$ and distance =
$(C/D)(\ln t)_{T+2}^{T+4}$
Since 2 p.m. to 4 p.m. distance = 0.5 × noon to 2 p.m. distance,
$(C/D)(\ln t)_{T+2}^{T+4} = 0.5 \times (C/D)(\ln t)_T^{T+2}$

Therefore, $\ln(T + 4) - \ln(T + 2) = 0.5 \times \ln(T + 2) - 0.5 \times \ln(T)$
$\ln(T + 4) + 0.5 \times \ln(T) = 1.5 \times \ln(T + 2)$
$\ln(T + 4)\sqrt{T} = \ln(T + 2)\sqrt{T + 2}$
$(T + 4)\sqrt{T} = (T + 2)\sqrt{T + 2}$

Squaring,
$T^3 + 8T^2 + 16T = T^2(T + 2) + 4T(T + 2) + 4(T + 2)$
$T^3 + 8T^2 + 16T = T^3 + 2T^2 + 4T^2 + 8T + 4T + 8$
$T^3 + 8T^2 + 16T = T^3 + 6T^2 + 12T + 8$
$2T^2 + 4T - 8 = 0$

Solving for T,
$T = [-4 \mp \sqrt{16 - 4(2)(-8)}]/4$
$= (-4 \mp \sqrt{80})/4 = 1.236068$ hours before noon, or at 10 o'clock, 44 minutes and 50+ seconds

Also solved by Richard Hess, Mary Lindenberg, Charles Sutton, Victor Newton, Norman Spencer, Pierre Heftler, Mike Hennessey, Ross Hoffman, Frank Carbin, Raymond Gaillard, Charles Rivers, Richmond Perley, Jerry Horton, Jacob Bergmann, Rik Anderson, Kelly Woods, Danny Mintz, Avi Ornstein, Gardner Perry, Sam Levitin, W. Gale Cutler, and the proposer, Bruce Calder.

Better Late Than Never

Jul 1. Dan Jones has responded.

Jul 4. Hal Vose has responded.

Proposers' Solutions to Speed Problems

SD 1. EIGHT.
Consider FIVE SIX SEVEN EIGHT and convert the Roman numerals to Arabic.

SD 2.

A dozen, a gross, and a score,
Plus three times the square root of four,
Divided by seven,
Plus five times eleven,
Equals nine squared plus not a bit more.