ALLAN J. GOTTLIEB

How to Computerize Your Engagements

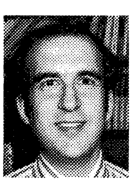
As promised in the October issue, here is the current backlog of submitted problems in various classes. The shortest delay until publication is for speed problems, where I have a half-year supply. The backlogs for chess, bridge, and regular problems are each approximately one year. Some confusion has developed concerning the computer-related problems. These problems will be clearly identified when presented, and I did not intend to suggest a preference for computer calculations over mathematical analyses for regular problems. To date (November 9), I have received two problems designed as computer-related, one of which appears as **F/M 1** below. Thus, the backlog for this class is just one problem.

Finally, I must apologize to Merle Smith for misspelling his name in the October column.

Problems

F/M 1. Alfred Anderson inaugurates the computer-oriented problems with this offering; he writes:

Recently I used brute force to solve a rather interesting computer-oriented problem. Perhaps one of your readers would find a more elegant solution. Management meetings are scheduled on the second Thursday of each month, administrative conferences are the third Friday, and work units have a seminar on the first Monday. Derive an algorithm which will generate a date given the year, month, day-of-week, and ordinal week within the month. For example, if a meeting were scheduled for the third Friday of August 1984, the algorithm would return "August 17." Note that a meeting on the fifth Tuesday in March would be fine for 1983, 1985, and 1986 but not for 1984 (there are only four Tuesdays in March of 1984). In this case the algorithm could return January 0.



SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, ASSOCIATE RESEARCH PROFESSOR AT THE COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y., 10012.

F/M 2. William Stein likes to deal with loosely coupled coins:

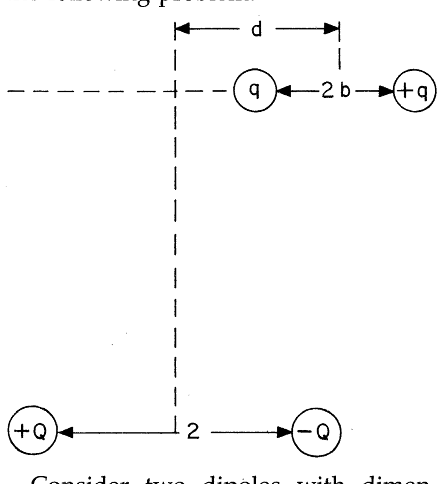
Two coins, loosely coupled, are flipped simultaneously such that if either one is heads, the other has probability 7/8 of also being heads, but if any one is tails, the other is equally likely to be either heads or tails. Find the probability of each individual coin turning up heads, and the probability of their both being heads simultaneously (or prove that the problem statement and data are inconsistent).

F/M 3. A geometry problem from Phelps Meaker:

A horizontal line of length 2a forms the common base for two isosceles triangles. On the near side the triangle is $45^\circ - 45^\circ - 90^\circ$, and on the opposite side $75^\circ - 75^\circ - 30^\circ$. Determine the radius of the circle tangent to all sides of the composite lanceolate figure, and locate its center.

F/M 4. Smith D. Turner $\int(dt)$ wants you to find a four-digit number whose square is an eight-digit number whose middle four digits are zero.

F/M 5. David Dreyfuss was attracted to the following problem:



Consider two dipoles with dimensions as indicated. The lower dipole is fixed, and the upper dipole is constrained to move along a horizontal line. (This is roughly the geometry encountered in magnetic stirring.) Find the conditions on h and b for which the upper dipole tends to center (the force is in the opposite direction of d, the displacement from the center line). When does the motion of the upper dipole approximate simple harmonic motion?

Speed Department

SD 1. David Evans has placed a turtle in each of the four corners of a square room measuring 3 meters on a side. All four start moving at the same instant at a constant speed of 1 cm./sec., and each crawls directly toward the turtle to the left. How long does it take for them to meet at the center of the room?

SD 2. Steven Bernstein knows a teacher who brings apples for her students:

Ms. Lang, the third-grade teacher, wanted to do something nice for the n students in her class. One day she brought in an apple which would be given to a lucky youngster. Her problem was how to choose the lucky one fairly. Here is what she decided to do: She said: "I'll pick a number from 1 to n. The first one of you to guess the number is the lucky winner. Let's hear your guesses in alphabetical order. Aaron, you're first. Be sure to speak up so everyone can hear you. Zelda, you'll be last." "Unfair!", said Aaron. "I'll have all those numbers to choose from. My chances of guessing are pretty small. Wanda, Yolanda, and Zelda will have a chance to hear everyone else's guesses so they will have a better chance of winning!" "Wait a minute," said Zelda. "I probably won't even get a chance to guess because all the other kids go first and one of them will win before it's even my turn!" Ms. Lang replied, "Children, I've thought about it and this procedure is fair. You all have the same chance of winning." Is Ms. Lang correct?

Solutions

OCT 1. What is the minimum number of high-card points needed to make a contract of 7 spades against the best defense?

Bob Sackheim sent us the following:

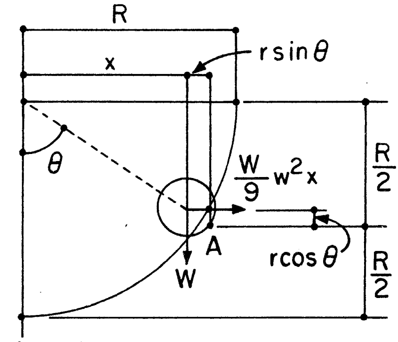
I suspect that the minimum number of high-card points needed to make a 7-spade contract is five. One possible deal would be as follows (point cards are shown; all other cards are indicated by x's, their values being immaterial):

	♠ x x x x x	
	♥ -	
	♦ x x x x x x x x	
	♣ -	
♠ K		♠ Q
♥ A Q x x		♥ K J
♦ K		♦ A Q J x
♣ A K x x x x x		♣ Q J x x x x
	♠ A J x x x x	
	♥ x x x x x x x	
	♦ -	
	♣ -	

If West leads the ♠K, South takes with the ♠A, cross-ruffs four hearts and four diamonds, setting up three hearts in his hand, plus his last spade for 13 tricks. If West leads a heart, North ruffs, leads a spade to the ♠A, then three more hearts and three more diamonds are cross-ruffed, setting up three good hearts in South's hand, plus his two remaining spades for 13 tricks. If West leads a diamond, South ruffs, plays the ♠A, then cross-ruffs four hearts and three diamonds, ending in dummy; his last spade, and either three hearts or three diamonds, are good. If West leads a club, South ruffs, plays his ♠A, cross-ruffs four hearts and four diamonds, ending in his hand; his last three hearts are good.

Also solved by Tom Harriman, Lester Steffens, Richard Boulay, Philip Dangel, Alan Robock, Warren Himmelberger, Winslow Hartford, and the proposer, Howard Sard.

OCT 2. Given a rotating hollow semi-sphere with a hole in the bottom, into which a marble is dropped, find the angular velocity required to hold the marble exactly halfway between top and bottom of the sphere. Assume the frictional coefficient between marble and sphere is μ and the radius of the sphere is R.



In figure shown,
r = radius of the marble
R = radius of the hemisphere
W = weight of the marble
w = angular velocity of the hemisphere
g = acceleration of gravity
and A is the point of contact between the marble and the spherical surface at a height R/2 above the bottom. For equilibrium in a vertical uniformly rotating plane, the moment of W about point A will equal the moment of the centrifugal inertia force $W/g \cdot w^2 x$ about the same point, Thus,
$Wr \sin\theta = W/g \cdot w^2 x \cdot r \cos\theta$, or
$w = \sqrt{g/x \cdot \tan\theta}$. Since
$\theta = \cos^{-1}(r/2)/R = 60°$,
then $\tan\theta = \sqrt{3}$. Also,
$x = (R - r)\sin\theta = \sqrt{3}/2 \cdot (R - r)$.
Substituting x and tan θ into equation (1) yields
$w = \sqrt{2g/(R - r)}$ rads./sec., the required angular velocity. If the intent of the problem was to have the center of the marble halfway above the bottom, we would have
$\cos\theta = R/[2(R - r)]$ and
$\sin\theta = x/(R - r)$. Hence, for this condition, the angular velocity from (1) would be:
$w = \sqrt{2g/R}$ rads./sec.

Also solved by Matthew Fountain.

OCT 3. In the Illinois Lottery Lotto game, the player chooses six different integers from 1 to 40. If the six match, in any order, the six different integers drawn by the lottery, the player wins the grand prize jackpot which starts at $1 million and grows weekly until won. Multiple winners split the pot equally. For each $1 bet, the player must pick two, presumably different, sets of six integers. Considering the grand prize alone, under what conditions would it pay, on the average, to play this game? In the game week ending June 18, 1983, 78 people matched all six winning integers and split the jackpot. Estimate the odds of this outcome, given that 2 million people bought $1 tickets that week.

There are $\binom{40}{6}$ or 3,838,380 possible combinations which might be chosen. It will prove highly convenient to round this off to 4×10^6. The expectation for a lottery participant would be easy to compute, if we didn't have to worry about multiple winners. Choosing 2 out of 4,000,000 combinations, someone would have a 1/2,000,000th chance of winning. The prize would have to exceed $2 million before it would pay, on the average, to play the game. But, in fact, the prize would have to be significantly higher than this, because there is a good chance of multiple winners. The probability that a winner will have to share his prize is only 1/e, as can easily be shown. There are, according to information given later in the problem, 2,000,000 people playing the game. That is, 4,000,000 combinations are selected. Since there are 4,000,000 *possible* combinations, the probability that any given selection does not match the winner (*assuming that each selection is equally likely*) is
$[(4 \times 10^6) - 1]/(4 \times 10^6)$.
The probability that all the other selections don't match the winner is
$([(4 \times 10^6) - 1]/4 \times 10^6)^4 \times 10^6$
This is very close to 1/e since, as is familiar from calculus,
$\lim [(n - 1)/n]^n = 1/e$.
So there is a probability of $1 - 1/e$ that there will be additional winners. We will need to know the probabilities of specific numbers of additional winners, and these can be obtained from the Poisson formula:
$\mu^k/k! \cdot e^{-\mu}$.
This gives the probability of k additional winners, where μ is the average number of winners. In the present case μ is 1, since there are 2,000,000 participants, each with a 1/2,000,000th chance of success. The following table may be compiled:

	Probability
No additional winner	.37 (= 1/e)
One additional winner	.37
Two additional winners	.18
Three additional winners	.06
Four additional winners	.01
Five or more winners	negligible

What this indicates is that a winner has a .37 chance of getting all the money, a .37 chance of getting half the money, etc. What the winner could expect would be, roughly,
$.34 + .37/2 + .18/3 + .06/4 + .01/5 = .63$
of the prize money. In order for the lottery to be a good bet, 63/100 times 1/2,000,000 times the prize money would have to exceed $1. That is, the prize money would have to exceed $3.17 million.

To determine the probability of the June 18, 1983, result, we again make use of the Poisson formula, with k = 78. This gives the probability that 78 people will have a given winning combination. Since 78! is on the order of 10^{115}, this is an extremely low probability. An event this unlikely could not, practically speaking, have happened. But it did happen. Therefore, something has gone wrong in the preceding calculations.

The problem lies in the assumption that each combination is equally likely to be picked by lottery participants. In fact, people do not choose numbers at random. They choose numbers which have some significance. Often this is a strictly individual significance (one's age, a date of the month with special meaning, etc.). But the choices are sometimes on the basis of a more general significance. Some numbers (e.g., 7 and, to a lesser extent, 3) are believed to be lucky. These and related numbers, such as 21, 33, and 37, will be heavily over-represented in the selections. By a reverse psychology, 13 might also be popular.

What *were* the winning numbers for June 18, 1983? This can be looked up in daily papers of the region for June 19. When I did look this up, I had some idea, based on the considerations in the last paragraph, of what I might be seeing. But, even so, I could hardly believe my eyes. The winning combination was: 7, 13, 14, 21, 28 and 35. It would be difficult to imagine a group more appealing to the believer in lucky numbers! Here we have all the products of 7 between 1 and 40, with 13 thrown in for good measure.

So the answer to the question "What was the likelihood of the June 18, 1983, result?" is now apparent, though it doesn't lend itself to exact quantification. It was *quite* remarkable (having a probability of $1/(4 \times 10^6)$ that that particular combination was the winner. It was *not at all* remarkable that 78 people chose it. The probability of the complex event (that number winning *and* being chosen by 78) is not much less than $1/(4 \times 10^6)$.

Let's consider the more general question of there being *some* result involving a very large number (say 50 or more) of winners. I am inclined to think that this is not extraordinarily unlikely. Take the following as a list (obviously not exhaustive or definitive) of lucky numbers between 1 and 40: 3, 7, 11, 13, 14, 21, 22, 28, 33, 35, 37. There are $\binom{11}{6} = 462$ combinations of these numbers which might be selected. Given that the total of possibilities is 4×10^6, it's not likely that any of these will again be selected in the foreseeable lifespan of the lottery. But consider the combinations of five of these numbers with some other number selected for its significance to an individual. There are $\binom{11}{5} \times 29$ (about 14,400) of these combinations. The chance that one of them will be a winner is roughly .004 (1 in 250). So one of these should come up about every five years, and when it does there may well be a lot of winners. Probably not as many as 78, for these combinations don't have the unique appeal of that other one, but 50 would not seem too unreasonable for some of these combinations.

It's now quite clear that my earlier calculation of what percentage of the prize a lottery player might expect to collect can only be interpreted as an average. What a particular player can expect will vary in relation to the numbers that are selected. If he picks two of the supposedly lucky combinations, his winnings may have to be shared with 50 others. If the grand prize is at the $1 million level, each winner's share is slightly less than $20,000. Recalling that there is still just a 1/2,000,000th chance of winning, this works out to an expectation of less than 1¢.

If you deliberately choose nothing but "ordinary" numbers, your expectation is somewhat improved beyond the level I gave earlier. Let's try a quite arbitrary assumption, in order to make the discussion more concrete. Suppose half of the lottery entrants make use (consciously) of one or more of the "lucky" numbers in picking their combinations. The other half choose their numbers on the basis of considerations peculiar to themselves, so that these selections are, in the aggregate, random. If you deliberately choose only "ordinary" numbers, then you are competiting only with the second group. There is no possibility that you will have to share your prize with anyone in the first group. Using the Poisson formula, we can compute a table of probabilities for this situation, similar to the table given earlier. But now μ is just 0.5. We have, on the supposition that you win:

	Probability
No other winners	.61
One other winner	.30
Two other winners	.08
Three other winners	.01

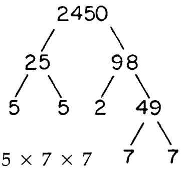
You would stand to receive $.61 + .30/2 + .08/3 + .01/4$, or about .79 of the prize. In the case where the prize is $1 million, this works out to an expectation of 40¢ (compared to the 1¢ expectation enjoyed by the other contestant). But note that under any circumstances, the prize must be significantly above $2 million before the lottery becomes a good bet.

Also solved by Frank Carbin, Warren Himmelberger, Matthew Fountain, and the proposer, Jonathan Hardis, who sent us a particularly complete solution. Mr. Hardis seems to be quite an authority on the Illinois lottery. Since he has a Chicago address, one might conjecture that some of his knowledge comes from empirical study.

OCT 4. An associate research professor walks into his office one morning and says to his secretary, "I had three dinner guests last night. The product of their ages was 2450. The sum of their ages was twice your age. Can you tell me their three ages?" Ten minutes later his secretary came to him and said that the problem could not be solved. He said, "You are right. I will now tell you that I was the oldest one there." The secretary was then able to tell him the ages of the three dinner guests. What are the ages of the dinner guests, her age, and the professor's age?

The following solution is from Tom Gallahan: The first thing to do is make a list of the possible combinations of ages. Once the secretary has done this he/she can eliminate the ones that do not add up to twice his/her age. If there were only one set of ages that fit the criteria it would be the answer. This is not the case because he/she cannot solve the problem. There must be more than one choice. The fact that the professor is the oldest one there must distinguish between the possible choices so that the secretary can solve the problem.

In making a list it helps to make a factor tree, since three ages's product is 2450.



$2450 = 2 \times 5 \times 5 \times 7 \times 7$

	Ages of guests	*Resulting age of secretary*
1	2, 5, 245	126
2	2, 7, 175	92
3	2, 25, 49	38
4	2, 35, 35	36
5	5, 5, 98	54
6	5, 7, 70	41
7	5, 10, 49	32
8	5, 14, 35	27
9	7, 7, 50	32
10	7, 10, 35	26
11	7, 14, 25	23

You must also realize that one or two of the guests may be one year old:

12	1, 25, 98	62
13	1, 35, 70	53
14	1, 49, 50	50
15	1, 1, 2450	1226

None of the guests may be 0 years old because the product of the guests' ages in that case will always be 0.

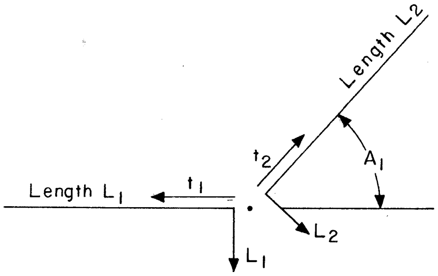
For ease of explanation I have numbered the sets of ages. All of the resulting secretary's ages are distinct except 7 and 9. One of these must be the correct answer; thus we now know that the secretary is 32 years old. In case 7 the professor must be 50 years old or older to be the oldest one there. In case 9 he must be 51 or older. If the professor is 51 or older the secretary can't choose between the sets. The professor must be 50 years old and 7 must be the correct set. Logically we could have eliminated 1, 2, and 15 but this is unnecessary.

Also solved by Michael Jung, Matthew Fountain, David Griesedieck, Tom Harriman, Naomi Markovitz, Richard Boulay, Fernando Saldanha, Jerry Sheldon, Clarence Cantor, Steve Feldman, Dennis Loring, Ronald Raines, James Michelman, Avi Ornstein, Tso Yee Fan, John Rosendahl, Raymond Gaillard, E.R. Foster, Danny Mintz, Roy Levitch, Miriam Nadel, Tom Lydon, Myles Friedman, Pierre Heftler, Frank Carbin, Winslow Hartford, Harry Zaremba, and the proposer, Merle Smith.

OCT 5. Given an irregular polygon of n sides, in which sequence should the sides be arranged and how should the corner angles be determined to give the greatest area?

The following solution is from the proposer, Irving Hopkins:

Consider that the polygon is the horizontal cross-section of a vertical open-topped water standpide, the rectangular sides of which are hinged together along their vertical edges. There is no friction in the hinges, and the lower end of the vessel rests in a friction-free manner on a horizontal plate with no leakage of water. Gravity causes the water to fall until the sides have moved to what must be the enclosure of maximum area. When movement of the water has ceased, the pressure at any depth is uniform in all directions. The stress in the sides depends on the depth, but the geometry does not vary with the pressure. Assume a depth at which a side of width L and a suitable vertical dimension is subject to a force 2L. At the junction of L_1 and L_2 the forces are as shown, where t_1 is the tension in L_1 and t_2 that in L_2.



At the hingepin, the forces parallel to L_1 balance if

$t_1 = t_2\cos A_1 + L_2\sin A_1$ (1)

and the perpendicular forces if

$L_1 + L_2\cos A_1 = t_2\sin A_1$ (2)

Going on to other corners, we have:

$t_2 = t_3\cos A_2 + L_3\sin A_2$ (3)

$L_2 + L_3\cos A_2 = t_3\sin A_2$ (4)

. . . . .

$t_n = t_1\cos A_n + L_1\sin A_n$ $(2n - 1)$

$L_n + L_1\cos A_n = t_1\sin A_n.$ (n)

From the even-numbered equations above:

$t_1 = (L_n + L_1\cos A_n)/\sin A_n$

$t_2 = (L_1 + L_2\cos A_1)/\sin A_1$

$t_n = (L_{n-1} + L_n\cos A_n - 1)/\sin A_1.$

Substituting these values of t in the odd-numbered equations, we get:

$(L_1\cos A_n + L_n)/\sin A_n = (L_1\cos A_1 + L_2)/\sin A_1$ (I)

$(L_2\cos A_1 + L_1)/\sin A_1 = (L_2\cos A_2 + L_3)/\sin A_2$

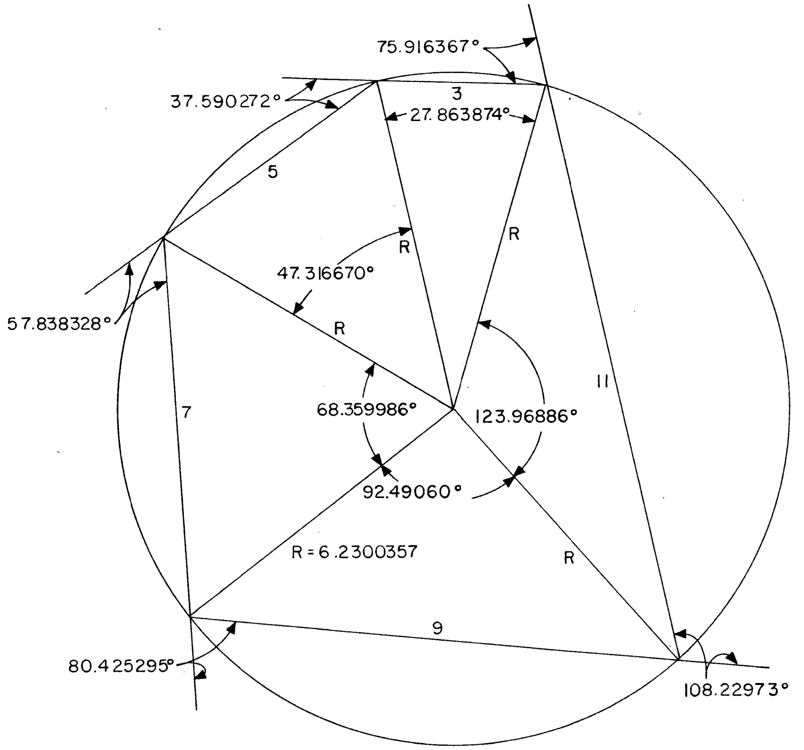
. . . . . (II)

$(L_n\cos A_{n-1} + L_{n-1})/\sin A_{n-1} = (L_n\cos A_n + L_1)/\sin A_n.$ (III)

We now have n equations from which to find the values of angles A_1 to A_n. But there is one more requirement: the sum of A_1 to A_n must be 360°. Assume a value for A_1 and let Q equal the left-hand side of (II). Square both sides of (II), which becomes

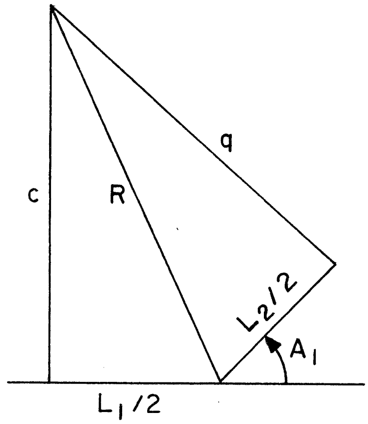
$\cos^2 A_2(L_2^2 + Q^2) + 2L_2L_3\cos A_2 + (L_3^2 - Q^2) = O.$

This is a quadratic equation in $\cos A_2$ from which A_2 may be found, and so on. For a pentagon with sides 3, 5, 7, 9 and 11 in length the angles were found to be



A_1	37.590°
A_2	57.838°
A_3	80.425°
A_4	108.230°
A_5	75.916°

Could such polygons be circumscribed by circles? If so, each side of the polygon is a chord of the circle, whose center must lie at the point of intersection of the perpendicular bisectors of the chords. Consider two adjacent sides of the polygons, say L_1 and L_2, with angle A_1.



Taking
$c = (L_2/2)\sin A_1 + q\cos A$, and horizontally
$L_1/2 = q\sin A_1 - (L_2/2)\cos A_1$.
From these equations we find that
$c = [L_2/2 + (L_1/2)\cos A_1]/\sin A_1$,
and the radius R_{12} determined from sides L_1 and L_2 is
$R_{12} = (c^2 + (L_1/2)^2)^{1/2}$
$= [(L_2/2) + (L_1/2)\cos A_1]^2/[\sin^2 A_1 + (L_1/2)^2]^{1/2}$
which boils down to
$4R_{12}^2 = (L_1^2 + 2L_1L_2\cos A_1 + L_2^2)/\sin^2 A_1$ (IV)
The existence of a circumscribing circle depends on $R_{12} = R_{23} = \ldots R_{n1}$. By analogy with (IV),
$4R_{23}^2 = (L_2^2 + 2L_2L_3\cos A_2 + L_3^2)/\sin^2 A_2$ (V)
Squaring both sides of (II), we get
$(L_1^2 + 2L_1L_2\cos A_1 + L_2^2\cos^2 A_1)/\sin^2 A_1 = (L_3^2 + 2L_2L_3\cos A_2 + L_2^2\cos^2 A_2)/\sin^2 A_2$. (VI)
Subtracting the left side of (VI) from the right side of (IV), and the right side of (VI) from the right side of (V), we get
$L_2^2(1 - \cos^2 A_1)/\sin^2 A_1 = L_2^2$, and
$L_2^2(1 - \cos^2 A_2)/\sin^2 A_2 = L_2^2$.
The right sides of (IV) and (V) are therefore equal; hence R_{12} and R_{23} and all the other R's are equal. The area of each triangle is easily found by
Area $= (L/2)[R^2 - (L/2)^2]^{1/2}$. The sequence in which the sides are arranged is immaterial, equivalent to carelessly cutting a pie and then shuffling the pieces.

The pentagon described is shown above. Each side subtends an angle at the center [$B_i = 2\arcsin(L_i/2R)$], and the sum of these must equal 360°. The easiest way to find the radius giving B = 360° is cut-and-try. This may fail if a very long side, L_n, causes the center of the circumscribing circle to be outside the enclosure. In this case, solve for

$$\sum_{i=1}^{n=1} B_i = B_n$$

Also solved by Matthew Fountain, Tom Harriman, Winslow Hartford, and Harry Zaremba.

Better Late Than Never

M/J 4. Andre Schmitz found a simpler way to present the solution.

A/S 4. Dick Swenson has responded.

Proposers' Solutions to Speed Problems

SD 1. Five minutes. Since the turtles move at right angles to each other, an approached turtle's motion does not contribute to the distance the approaching turtle must travel.

SD 2. Yes, Ms. Lang is correct. Each child has probability 1/n of winning. Aaron's chance is 1/n because he has n numbers to choose from. The k-th child will win if each of the k − 1 children that went before him/her guessed incorrectly and the k-th guesses correctly. The probability of this happening is:
$[(n - 1)/n][(n - 2)/(n - 1)] \ldots (n - k + 1)/(n - k + 2)[1/(n - k + 1)] = 1/n$.

Notice that in order for this to work it is necessary for each guess to be heard by the children to follow. In this way the disadvantage of going near the end is exactly compensated by a narrowing down of choices.