ALLAN J. GOTTLIEB

100 Ways to Say 1985

This being the first issue of another year, we again offer a "yearly problem" in which you are to express small integers in terms of the digits of the new year (1, 9, 8, and 5) and the arithmetic operators. The problem is formally stated in the "Problems" section, and the solution to the 1984 yearly problem is in the "Solutions" section.

Problems

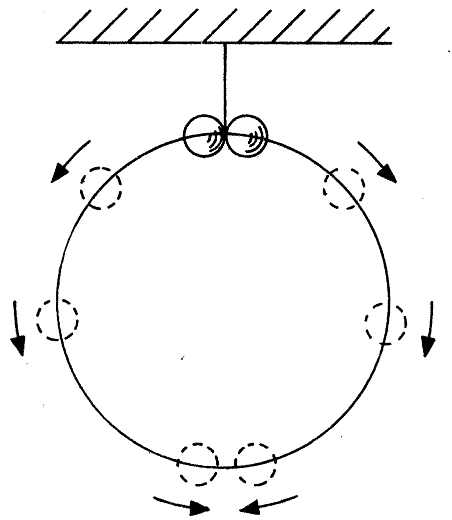
Y 1985. Form as many as possible of the integers from 1 to 100 using the digits 1, 9, 8, and 5 exactly once each and the operators +, −, × (multiplication), / (division), and exponentiation. We desire solutions containing the minimum number of operators; and, among solutions having the minimum number of operators, those using the digits in the order 1, 9, 8, and 5 are preferred. Parentheses may be used for grouping; they do not count as operators.

JAN 1. Our next problem is the last member of Emmet Duffy's collection of seven-card bridge problems. For the current challenge, South is on lead and is to take six tricks against best defense with hearts as trump:

	♠ 9 8 3 2 ♥ 10 ♦ A Q ♣ —	
♠ J ♥ 8 ♦ J 10 9 8 ♣ 5		♠ 10 ♥ 6 ♦ K 6 ♣ K J 6
	♠ A K 7 ♥ — ♦ — ♣ A Q 4 3	

JAN 2. Bruce Calder, after working on **1983 N/D 4,** sent us the following spin-off, a problem demonstrating the elegant subtlety of Newtonian mechanics:

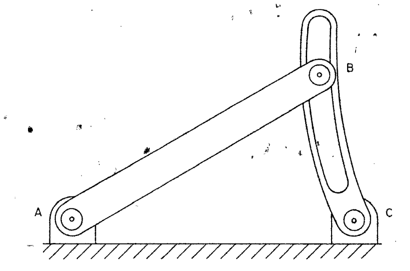
A smooth, rigid, and circular wire hoop hangs from a rigid support by an ideal, extensionless string. Two small beads slide along the hoop (like beads of a necklace) with negligible drag and friction. The beads are slid to the top of the hoop and released. How massive must each bead be to spontaneously lift the hoop?



JAN 3. Here's one John Rule dug out of the file where he keeps interesting problems encountered from various sources:

A man received a check calling for a certain amount of money in dollars and cents. When he went to cash the check, the teller made a mistake and paid him the amount which was written in cents in dollars, and vice-versa. Later, after spending $3.50, the man suddenly realized that he had twice the amount of money the checked called for. What was the amount on the check?

JAN 4. Our last regular problem is from Floyd Kosch:



SEND PROBLEMS, SOLUTIONS, AND COMMENTS TO ALLAN J. GOTTLIEB, '67, ASSOCIATE RESEARCH PROFESSOR AT THE COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, N.Y., 10012.

A rigid arm pivots around the fixed point A. At the end of the arm is a follower (B) which runs in a curved track. The track pivots about the fixed point C. If AB = AC = r, find the shape of the track such that its slope at C is always vertical.

Speed Department

SD 1. Here's one from Smith D. Turner (∫dt):

Bill and Joe are to be paid $10 to wrap and address a pile of packages. Joe addresses one while Bill wraps one, but Bill addresses three while Joe wraps one. How should the $10 be divided between them?

SD 2. We end with a bridge quickie from Doug Van Patter:

North:	**South:**
♠ J 6	♠ A K 7 4
♥ A Q 5	♥ K J 3
♦ A 2	♦ K J 5 4
♣ A Q 9 8 6 2	♣ K 5

You are declarer (South) in a six-no-trump contract. The opening lead of the ♦10 is taken with your ♦J. Now you wish you were in a grand slam. When you cash the ♣K, West shows out. Can you find any reasonable chance of still making your contract?

Solutions

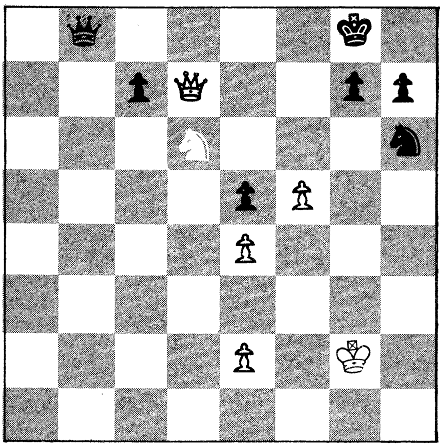
Y 1984. This is the same problem as Y 1985 (see above) with only the one digit changed.

The following solution is from Harry Zaremba; he points out that 36 numbers, shown with asterisks, use the digits in the same order as the year 1984:

*1 = 1^{984}
2 = 4/(18/9)
*3 = 1^9 + 8/4
*4 = 1 − 9 + 8 + 4
*5 = 1 × 9 − 8 + 4
*6 = 1 + 9 − 8 + 4
*7 = 19 − 8 − 4
*8 = 1 + 9 − 8/4
9 = 9 × 1^{84}
10 = 9 − 1 + 8/4
*11 = 1 × 9 + 8/4
*12 = 1 + 9 + 8/4
13 = 94 − 81
*14 = 1 + 9 + 8 − 4
*15 = 19 − 8 + 4
16 = (9 − 1) × 8/4
*17 = 19 − 8/4
*18 = 1 × 9 × 8/4
*19 = 1 + 9 × 8/4
*20 = (1 + 9) × 8/4
*21 = 19 + 8/4
*22 = 1 + 9 + 8 + 4
*23 = 19 + 8 − 4
24 = 41 − 9 − 8
25 =
26 =
27 = 9 × 4 − 8 − 1
28 = 9 × 4 − 8 × 1
29 = 48 − 19
30 =
*31 = 19 + 8 + 4
32 = 81 − 49
*33 = 1^9 + 8 × 4
34 =
35 = 9 × (4 − 1) + 8
36 = 81/9 × 4
37 = (1 + 4) × 9 − 8
*38 = 19 × 8/4
39 = 48 − 9 × 1
40 = 41 − 9 + 8
*41 = 1 × 9 + 8 × 4
*42 = 1 + 9 + 8 × 4
43 = 91 − 48
*44 = (19 − 8) × 4
45 = 81 − 9 × 4
46 =
47 = 48 − 1^9
48 = 89 − 41
49 = (1 + 4) × 8 + 9
50 = 49 + 1^8
*51 = 19 + 8 × 4
52 =
53 = (1 + 4) × 9 + 8
54 = 18 + 9 × 4
55 =
56 = 48 − 1 + 9
57 = 98 − 41
58 = 41 + 9 + 8
59 = 91 − 8 × 4
60 = (9 − 1) × 8 − 4
61 =
62 =
63 = 18 × 4 − 9
64 = (9 + 8 − 1) × 4
65 = 84 − 19
66 =
67 = 48 + 19
*68 = 1 × 9 × 8 − 4
*69 = 1 + 9 × 8 − 4
70 =
71 = 9 × 8 − 1^4
*72 = (1 + 9 + 8) × 4
73 = 9 × 8 + 1^4
74 = 84 − 9 − 1
75 = 89 − 14
*76 = 1 − 9 + 84
*77 = 1 + 9 × 8 + 4
78 =
79 = 91 − 8 − 4
80 = (1^4 + 9) × 8
*81 = 1 × $9^{(8/4)}$
*82 = 1 + $9^{(8/4)}$
83 = 84 − 1^9
84 = 98 − 14
*85 = 1^9 + 84
86 = 81 + 9 − 4
87 = 91 − 8 + 4
88 = 89 − 1^4
89 = 91 − 8/4
90 = 18 × (9 − 4)
91 = (9 + 4) × (8 − 1)
92 = 89 + 4 − 1
*93 = 1 × 9 + 84
*94 = 1 + 9 + 84
*95 = 1 + 98 − 4
96 = (9 + 4 − 1) × 8
97 = 98 − 1^4
98 = 98 × 1^4
99 = (8 + 4 − 1) × 9
100 =

Also solved by Avi Ornstein, George Aronson, Harry (Hap) Hazard, Phelps Meaker, Joe Feil, Peter Silverberg, Allan Tracht, and A. Holt.

A/S 1. Given the situation shown, White to move and win.



Bert Daniels had a little trouble with this one:
1. N-B8, K-R1 (1. . . ., P-N3 loses to 2. P-B6, N-B2, 3. Q-K8 mate; while 1. . . ., K-B1 loses to 2. Q-K7 ch and Q-K8 mate);
2. Q-Q8 ch, N-N1;
3. N-Q6 and the threat of smothered mate wins the queen.

Also solved by Matthew Fountain, Kenneth Bernstein, R. Hess, Avi Ornstein, David Evans, David Detlefs, George Aronson, Ronald Raines, Philip Dangel, and the proposer, Robert Kimble.

A/S 2. An ordinary combination padlock requires three ordered numbers to open, each between 0 and 39, inclusive. Thus there are 64,000 possible combinations. If it is known that the sum of the three numbers is 58 and the sum of the individual digits of all three numbers is 13, how many combinations are possible? If each of these possible combinations is equally likely, what is the probability that the second number is 34?

Many readers submitted computer programs that calculated all possibilities. I preferred analyses that reduced the number of possibilities to a manageable level. Matthew Fountain actually submitted both a program and an analysis. Here is the latter:

Let A equal the sum of the tens digits of the three ordered numbers and B equal the sum of the units

	Permutations	
3,2,0	6	(2)
3,1,1	3	(1)
2,2,1	3	
Total	12	(3)

( ) indicates number of permutations with 3 in second position.

	Permutations	
8,0,0	3	
7,1,0	6	
6,2,0	6	
6,1,1	3	
5,3,0	6	
5,2,1	6	
4,4,0	3	(2)
4,3,1	6	(2)
4,2,2	3	(1)
3,3,2	3	
Total	45	(5)

( ) indicates number of permutations with 4 in second position.

Number of sides	Edges each side	Total edges	Number of dihedrals	Dihedrals meeting at each apex	Number of apices
4	3	12	6	3	4
6	4	24	12	4	6
8	3	24	12	4	6
12	5	60	30	3	20
20	3	60	30	5	?

digits of the three ordered numbers. Then $A + B = 13$ and $10A + B = 58$, with solutions $A = 5$ and $B = 8$. The table at the bottom of page A25 shows that A can be decomposed into three ordered digits, none exceeding 3, in twelve ways, and B can be decomposed into three ordered digits 45 ways. The total combinations are $12 \times 45 = 540$. Those with 34 as second number are $3 \times 5 = 15$. The probability that 34 is the second number is $15/540 = 1/36$.

Also solved by Kenneth Bernstein, Richard Hess, Avi Ornstein, David Evans, David Detlefs, George Aronson, Dennis Sandow, Rita Carp, Gerry Grossman, Harry Zaremba, Richard Marks, Winslow Hartford, Yale Zussman, P. Jung, Frank Carbin, Steve Feldman, Aaron Hirschberg, Dave Mohr, Alfred Anderson, Thomas Stowe, and the proposer, John Prussing.

A/S 3. Fill in the missing entry in the table above pertaining to regular polyhedra. Can the values in the last column be determined by a formula?

The table contained two typos: a square has eight apices and three dihedrals meeting at each apex. These errors did not seem to cause much trouble. In particular, Avi Ornstein submitted the following:

The missing number is 12 apices for the icosahedron. The number of apices is given by the following:

(number of sides)(edges on each side)/(dihedrals meeting at each apex)

or

(total edges)/(dihedrals meeting at each apex)

or

2(number of dihedrals)/(dihedrals meeting at each apex).

Also solved by Matthew Fountain, Kenneth Bernstein, Richard Hess, David Evans, David Detlefs, Gerry Grossman, Harry (Hap) Hazard, Harry Zaremba, Richard Marks, Winslow Hartford, Yale Zussman, and Albert Mullin.

A/S 4. Find infinitely many positive integers n not containing the digit zero such that $n^2 - 1$ contains just two digits neither of which is zero. The digits may be repeated.

Jerry Marks sent us three patterns:

7	48	4	15
67	4488	34	1155
667	444888	334	111555
6667	44448888	3334	11115555
66667	4444488888	33334	1111155555
5	24		
65	4224		
665	442224		
6665	44422224		
66665	444422224		

Richard Hess has a proof of one of these patterns:

$(2 \times 10^n/3 + 1/3)^2 - 1 = 4 \times 10^{2n}/9 + 4 \times 10^n/9 + 1/9 - 1$

$= 4(10^{2n} - 1)/9 + 4(10^n - 1)/9 + 4/9 + 4/9 - 8/9$

$=$ 2n 4's + n 4's

$=$ n 4's followed by n 8's.

Also solved by Kenneth Bernstein, David Evans, David Detlefs, Dennis Sandow, Gerry Grossman, Harry Zaremba, and the proposer, Matthew Fountain.

A/S 5. Given a triangle ABC, draw its incircle and consider triangle DEF determined by the points of tangency. Show that the area of triangle DEF is $(r/d)A$, where r is the radius of the incircle, d is the diameter of the circumcircle, and A is the area of triangle ABC.

We give two different solutions, the first from Kenneth Bernstein and the second from Phelps Meaker:

Bernstein begins by letting the sides of triangle ABC be a, b, and c. Define k by:

$k = \sqrt{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}$

The radius, R, of the circumcircle is abc/k. The area of triangle ABC is $k/4$. The radius, r, of the incircle is $k/2(a + b + c)$. Denote the side FE of triangle FED by a'. Then a' can be expressed in terms of r and angle A:

$(a')^2 = 2r^2(1 + (\cos A))$.

The term $(\cos A)$ can be expressed in terms of a, b, and c using the law of cosines:

$a^2 = b^2 + c^2 - 2bc \cos A$.

Combining the last two expressions:

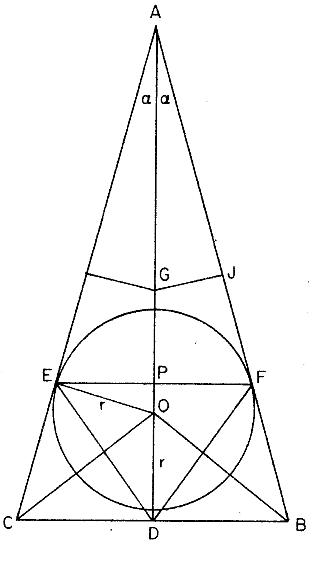
$a' = (-a + b + c)/2 \times \sqrt{(a - b + c)(a + b - c)/bc}$,

with similar expressions for b' and c'. The area of triangle FED is $k'/4$ where k' is defined similarly to k with a' substituted for a, etc. After much algebra, the area of triangle FED is

$k(-a + b + c)(a - b + c)(a + b - c)/16abc$

$= k^3/16abc(a + b + c)$

$= (r/2R)(\text{area ABC})$.



To establish his solution, Phelps Meaker lets H equal the altitude AD of triangle ABC; O is the center of the incircle; G is the center of the circumcircle; and α is one-half of the apical angle. Then, with respect to triangle AEO:

$r = EO = OD = (H - r) \sin \alpha$;

$r + r \sin \alpha = H \sin \alpha$; and

$r = H \sin \alpha/(1 + \sin \alpha)$.

With respect to triangle ABD:

$AJ = AB/2 = H/2 \cos \alpha$;

$AG = AJ/\cos\alpha = H/2\cos^2\alpha$;
$d = 2AG = H/\cos^2\alpha$.
The area of triangle ABC is
$H \times H \tan\alpha = H^2 \sin\alpha/\cos\alpha$.
With respect to triangle EDF:
$EP = EO\cos\alpha$; angle $PEO = \alpha$; $OP = EO\sin\alpha$.
Then the area of triangle EDF is given by
$EP \times (OD + OP) = r\cos\alpha\,(r + r\sin\alpha)$
$= r^2\cos\alpha + r^2\sin\alpha\cos\alpha = r^2\cos\alpha\,(1 + \sin\alpha)$.
Substituting for r, the area of triangle EDF is given by
$H^2\sin^2\alpha\cos\alpha\,(1 + \sin\alpha)/(1 + \sin\alpha)^2$
$= H^2\sin^2\alpha\cos\alpha/(1 + \sin\alpha)$.
$(r/d)A = [H\sin\alpha/(1 + \sin\alpha)][\cos^2\alpha/H][H^2\sin\alpha/\cos\alpha]$
$= H^2\sin^2\alpha\cos\alpha/(1 + \sin\alpha)$.

Also solved by Matthew Fountain, Richard Hess, David Evans, Robert Hollenbach, Henry Lieberman, Mary Linderman, Howard Stern, and the proposer, Harry Zaremba.

Better Late Than Never

1983 JUL 5. Matthew Fountain sent us the following:
Donald Savage's comments on **1983 JUL 5** stimulated me to further investigation. I found (a) there are $29(2)^n$ n-digit numbers whose squares are suitable with respect to their last n digits and (b) there are $29(2)^n(0.2)^r$ n-digit numbers whose squares are suitable with respect to their last $n + r$ digits; (a) is exact when $n>3$; (b) is an excellent approximation when n is large and r small. Both (a) and (b) are consistent with the notion that the middle digits of squares are representative of random numbers, except that (a) is more than chance. I wrote a Pascal program that generated all the (a) numbers up to and including those of 20 digits and printed out those that came closest to having entirely suitable squares. My IBM took about 2.5 days to cover the range of 15 through 20 digits. I found that the results differed from (b) because of the pecularity of two sequences of digits. For example, (b) expects there to be 3.1 numbers of 20 digits or less with squares having their last 30 digits suitable. Actually there are 22. But 15 differ from 83,333,333,333,333,333,332 only in the first six or less digits, and four differ from 21,666,666,666,666,666,662 in the first five or less digits. Only 78,537,356,970,849,674,736 appears to resemble a random number. My conclusion is that (b) is probably a good estimate of the odds that there exists a large n-digit number with an $(n + r)$-digit square, all of whose digits are suitable. The pecularity of the two sequences should not affect (b) when $r = n$ or $r = (n - 1)$. The odds, of course, seem very small. It is interesting that Los Alamos in its early days took sequences of digits from the middles of consecutive squares as random numbers, a suggestion of von Neumann. Some cyclic patterns were observed, the worst being too many consecutive zeroes.

1984 APR 1. Harry (Hap) Hazard points out that spades can be led or steel but not lead.

APR 2. Richard Halloran has responded.

M/J 5. Dayton Datlowe has responded.

JUL 2. Ivor Morgan and Mary Lindenberg have responded.

A/S SD2. Arthur Carp, Ronald Martin, and Gordon Thomas report that pan's dimensions are $7 \times 5 \times 1.5$ inches.

Proposers' Solutions to Speed Problems

SD 1. The editor needs to assume that the speed ratio between Joe and Bill is constant for wrapping and addressing. In that case, the ratio is $\sqrt{3}$ and Joe should receive
$\$10/(1 + \sqrt{3})$.

SD 2. Lead low toward the ♠J, and hope West has the ♠Q. (My partner found this line of play, but most declarers went down.)