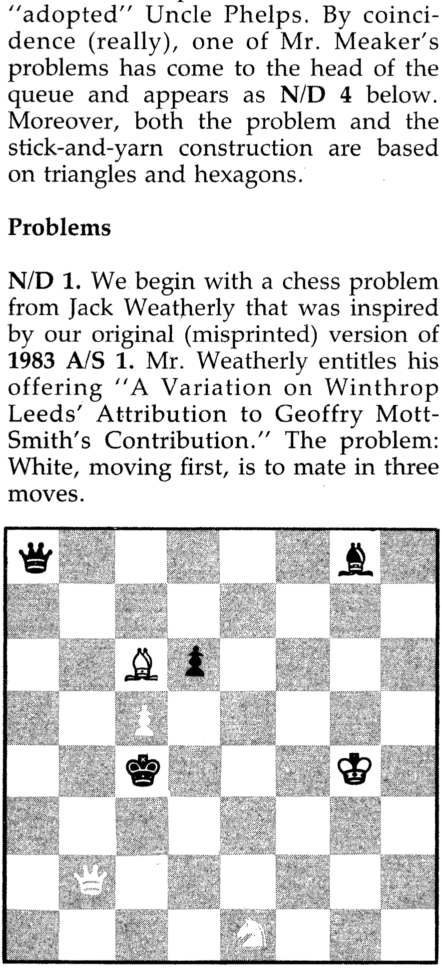
ALLAN J. GOTTLIEB

The Life Expectancy of a Moolah Dola

Phelps Meaker, '25, *Puzzle Corner*'s most faithful 83-year-old contributor, often supplements his problems and solutions with best wishes for our son David. Now he has sent David a beautiful handmade rendition of the star of David constructed out of six radial sticks and many strands of yarn connecting alternate radii. It has quickly become one of the family's prized possessions. Since I have no siblings and Alice but one, David doubled his uncle quota now that we have "adopted" Uncle Phelps. By coincidence (really), one of Mr. Meaker's problems has come to the head of the queue and appears as **N/D 4** below. Moreover, both the problem and the stick-and-yarn construction are based on triangles and hexagons.

Problems

N/D 1. We begin with a chess problem from Jack Weatherly that was inspired by our original (misprinted) version of **1983 A/S 1.** Mr. Weatherly entitles his offering "A Variation on Winthrop Leeds' Attribution to Geoffry Mott-Smith's Contribution." The problem: White, moving first, is to mate in three moves.



N/D 2. Smith D. Turner (∫dt) wants you to find a number that equals its own logarithm.

N/D 3. Here is a problem that appeared a few years ago in the annual M.I.T. Math Club contest:

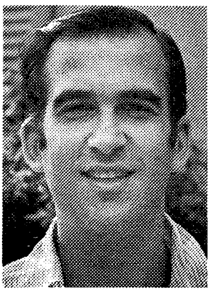
The floor lamps found in many Institute rooms pose an interesting problem. Each of these lamps has two bulbs, and each socket has a chain which when pulled will change the on/off state of only the bulb in that socket. When the lamp is on, it is difficult to determine whether one (and if so which one) or both bulbs are on. The problem is to find the shortest sequence of pulls that will turn the lamp completely off sometime during the sequence (e.g., if chains are labeled A and B, AABBAB fits the requirements, but a shorter sequence may be found). Can you generalize your solution to a lamp of three bulbs, four bulbs, etc.? Is your solution unique?

N/D 4. A geometery problem from Phelps Meaker:

A regular hexagon can be inscribed in an equilateral triangle so that its alternate sides coincide with the sides of the triangle. What is the ratio between the areas of the hexagon and the triangle?

N/D 5. Frank Rubin's money (and problem stockpile) never seem to run out:

In the country of Moolah, the national bank issues new dola bills to replace each bill that wears out or is lost or destroyed, so there is always a constant number of dolas in circulation. On January 1, the bank issued a new bill with the picture of Prince Centime replacing that of the late Queen Peseta. After one year, they found that 10/27 of the bills in circulation were the new variety. After two years, 2/3 of the bills; after three years 8/9 of the bills; and after four



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years all the bills were the new type. What is the life expectancy of a dola bill?

Speed Department

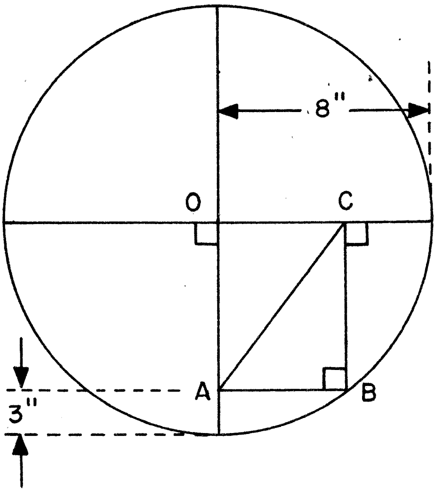
SD 1. A bridge quickie from Doug Van Patter:

North (Dummy):
♠ 9 6 2
♥ K 4 3
♦ A 8
♣ A 9 7 6 2

East (You):
♠ K Q J 8 3
♥ J
♦ K 10 5
♣ K 8 4 3

You (East) are defending against a six-heart contract in a high-stakes game. Your partner leads the ♣5, which is taken by the ♣A. Declarer plays three rounds of trumps, then leads the ♣Q to your ♣K. Your partner shows out. Can you find a way to save a lot of money?

SD 2. Chris Wee wants you to find the length of AC:



Solutions

JUL 1. You are South and open one no-trump. Your partner falls in love with her hand, transfers in spades, and puts you in a highly optimistic six-spade contract. Can you justify her faith in your declarer play?

North:
♠ J 10 9 8 6 3
♥ A J 5
♦ K
♣ A 9 3

West:
♠ Q 4
♥ K 7 3
♦ 10 7 6 4
♣ Q J 10 7

East:
♠ 7 5 2
♥ Q 9 4 2
♦ Q 5 3 2
♣ 5 4

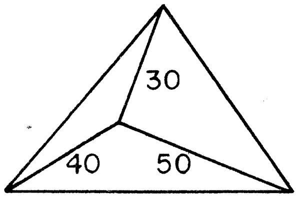
South:
♠ A K
♥ 10 8 6
♦ A J 9 8
♣ K 8 6 2

Opening lead: ♣Q.

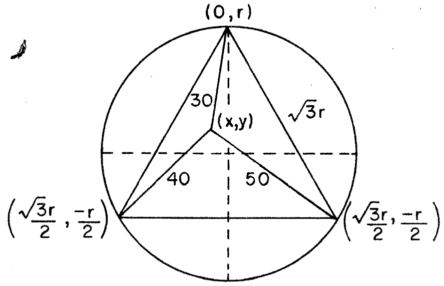
Rev. Joseph Hahn sent us the following solution: Win the club in dummy. Play the ♦K. Small trump to hand. Ruff a small diamond. Another small trump to hand. Ruff another diamond. Small club to ♣K. Play ♦A, throwing off the last club. Lead small heart to finesse West. If he covers, play ♥A, then ♥J, and the rest are yours. If he ducks, play ♥J. East must lead a heart through your 10-8 to pick up West's ♥K. (If he plays small, put in the 8; if he plays the ♥9, put in the ♥10.)

Also solved by Joseph Lambert, Carl Estes, G. Zartarian, Matthew Fountain, Richard Hess, Winslow Hartford, John Bobbit, and the proposer, Doug Van Patter.

JUL 2. Given an equilateral triangle with the indicated lines and lengths, determine the length of a side.



Norman Spencer remarks that he has seen this problem many times in the past 30 years including once in the 1979 June/July issue of *Puzzle Corner* (oops). The following solution is from Howard Stern:



First circumscribe the triangle and orient it as shown above. The circle has radius r and the side of the triangle, by simple geometry, has length $\sqrt{3}r$. The coordinates of the interior point where the three lines meet are (x,y). By the Pythagorean Theorem we have:

$$x^2 + (y - r)^2 = 30^2 \quad (1)$$
$$(x - \sqrt{3}r/2)^2 + (y + r/2)^2 = 50^2 \quad (2)$$
$$(x + \sqrt{3}r/2)^2 + (y + r/2)^2 = 40^2 \quad (3)$$

Expanding, then simplifying the above gives:

$$x^2 + y^2 + r^2 - 2yr = 900 \quad (1)$$
$$x^2 + y^2 + r^2 + yr - \sqrt{3}xr = 2500 \quad (2)$$
$$x^2 + y^2 + r^2 + yr + + \sqrt{3}xr = 1600 \quad (3)$$

Subtract (3) from (2) to get:
$x = -450/(\sqrt{3}r)$.
Add (2) and (3) and subtract twice (1) to get:
$y = 1150/(3r)$.
Substitute these results into (1) to get an equation in r:
$9r^4 - 15000r^2 + 1930000 = 0$.
Using the quadratic formula to solve for r^2, we get:
$r^2 = 2500/3 + 400\sqrt{3}$.
The side of the equilateral triangle is $\sqrt{3}r$ or:
$\sqrt{3}[2500/3 + 400\sqrt{3}]^{1/2} = 67.66432567$.
In addition, the Law of Cosines can be used to prove that the angle formed by the 30 and 40 lines is exactly 150°.

Also solved by Matthew Fountain, Winslow Hartford, Richard Hess, Judith Longyear, A.C. Lawson, David Parker, Richard Heldenfels, Harry Zaremba, Roy Boyle, Avi Ornstein, Mel Garelick, Norman Wickstrand, Clarence Cantor, Everett Leedy, Steve Feldman, Henry Lieberman, Peter Card, Frank Carbin, F.C. Jelen, David Evans, Joe Lovington, Eugene Sard, and Farrel Powsner who noticed that a similar problem was used by the New York City Interscholastic Mathematics League.

JUL 3. If a chess cube is placed in the corner square of a 3 × 3 chessboard, there are six distinct shortest paths (i.e., combinations of vertical and horizontal moves which do not backtrack) to the diagonally opposite corner. How many distinct shortest paths exist from a corner cell of a 3 × 3 × 3 cube to the cell opposite along a space diagonal, assuming movement parallel to the three axes? How about in a 3 × 3 × 3 × 3 hypercube?

Charles Sutton makes it look easy:
Assuming the shortest paths in the 3 × 3 chessboard are from the lower left to the upper right corner, the six paths can be designated RRUU,

RURU, RUUR, URRU, URUR, and UURR, where R and U denoted unit moves to the right and up, respectively. The number of such paths is clearly the number of four-letter "words" that can be formed using two R's and two U's. If the two R's are first placed in the four blank spaces, - - - -, the number of ways of doing this is the number of combinations of four things taken two at a time, which is just the binomial coefficient:

$\binom{4}{2} = (4 \cdot 3)/(1 \cdot 2) = 6.$

In a 3 × 3 × 3 cube, starting from the bottom left front corner, each shortest path can be represented by a six-letter word formed using two R's, two U's, and two B's (right, up, and back). We now have six blank spaces to fill, - - - - - -. If the two R's are placed first, the number of ways of doing this is the number of combinations of six things taken two at a time:

$\binom{6}{2} = (6 \cdot 5)/(1 \cdot 2) = 15.$

But for each of these placements, there are four remaining spaces to be filled by two U's and two B's, and, as in the 3 × 3 case, this can be done in six ways. Thus the number of shortest paths is 15 × 6 = 90. Now for the 3 × 3 × 3 × 3 hypercube. If we let H be the hyperspace direction, we have to figure how many eight-letter words can be formed using two each of the letters R, U, B and H. The number of ways the two R's can be placed in the eight blank spaces is:

$\binom{8}{2} = (8 \cdot 7)/(1 \cdot 2) = 28.$

But for each of these placements, there are six remaining spaces to be filled by two each of the letters U, B, and H, which can be done in 90 ways, as in the 3 × 3 × 3 case. Thus the number of shortest paths in the 3 × 3 × 3 × 3 hypercube is (28)(90) = 2520. Continuing this process, it is easy to see that the number of shortest paths in an n-dimensional cube, three units on a side, will be:

$\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2} \cdots \binom{4}{2},$

which can be simplified to $(2n)!/2^n$. And, using the same procedure, the number of shortest paths in an n-dimensional cube which is r units on a side turns out to be

$[n(r-1)]!/[(r-1)!]^n$.

Also solved by Matthew Fountain, John Bobbit, Richard Hess, Dick Wingerson, and the proposer, David Evans.

JUL 4. Show that it is physically impossible to "load" a pair of ordinary dice such that each of the eleven possible rolls is equally likely to occur. (Generalize to the other four regular polyhedra—e.g., the possible rolls of a pair of regular dodecahedra).

Oren Cheyette informs me that by solving **JUL 4** (the solution we are reprinting below) and **JUL 5** he has extended his anticipated Ph.D. completion date by one day. Here is one of those costly solutions:

For a pair of k-sided dice there are 2k − 1 possible rolls, from 2 to 2k. For "uniform" dice, the probability of each result must be 1/(2k − 1). Denote the probability of rolling an n on the first die p_n and on the second die q_n, and the probability of a total m on the two dice by P_m. Then

$P_2 = p_1q_1 = P_{2k} = p_kq_k = 1/(2k-1)$

while

$P_{k+1} = p_1q_k + p_kq_1 \ldots = 1/(2k-1)$

where . . . denotes the positive probability of getting a (k + 1) some way other that with a k and a 1. But

$q_1 = [1/(2k-1)]/p_1$ and

$q_k = [1/(2k-1)]/p_k$, and thus

$p_{k+1} = [1/(2k-1)][p_1/p_k + p_k/p_1] + \ldots$.

Now $x + 1/x > 2$ for positive x, so

$P_{k+1} > 2[1/(2k-1)]$, contradicting the requirement $P_{k+1} = 1/(2k-1)$.

Also solved by Winslow Hartford, Richard Hess, Henry Lieberman, Peter Card, Pierre Heftler, Tony Trojanowski, and the proposer, Albert Mullin.

JUL 5. A man gambles $100 a night by betting on black at roulette. Usually he places a single $100 wager and quits for the night. Since the wheel has only a single zero and pays even money, he has an 18/37 chance of winning $100. Suppose, instead, he starts with $100 but makes $1 bets until he runs out of money or is up $100. What is the probability that he will be a winner after a night of gambling?

Several readers used gambler's ruin formulas or Markov chains to solve this problem. I selected Matthew Fountain's solution since he simply established a clearly valid recurrence and plugged in the natural boundary conditions. However, Mr. Fountain doubts that the better very often completes his betting in one night. If he loses at the rate of $1 per 37 spins, he must average about 3700 plays to lose $100. This would require more than five spins per minute for twelve hours. Incidentally, Winslow Hartford believes somebody should tell Mr. Jones that it is at Monte Carlo that there is only one zero. Las Vegas pays for the floor shows with an extra zero, and the odds are 18/38.

Also solved by Frank Carbin, F.C. Jelen, David Evans, Steve Feldman, Oren Cheyette, Richard Hess, John Prussing, Ronald Raines, and the proposer, Richard Jones.

Better Late Than Never

M/J 1. Ronald Raines has responded.

Proposers' Solutions to Speed Problems

SD 1. Put the ♦K on the table. This may lose a diamond trick but guarantees that declarer cannot pitch losers on dummy's last two clubs. (Declarer's ♣J is still blocking the suit. Declarer's club holding was originally Q J 10.)

SD2. 13 inches.

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