ALLAN J. GOTTLIEB

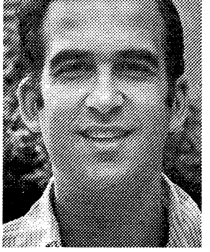
The King of Marbles Takes a Queen

Since this is the first issue of a new academic year. I once more review the ground rules under which this department is conducted.

In each issue I present five regular problems (the first of which is chess- or bridge-related) and two "speed" problems. Readers are invited to submit solutions to the regular problems, and three issues later one submitted solution is printed for each problem; I also list other readers whose solutions were successful. For example, solutions to the problems you see below will appear in the February/March issue. Since I must submit that column sometime in November (today is July 20), you should send your solutions to me during the next few weeks. Late solutions, as well as comments on published solutions, are acknowledged in the section "Better Late Than Never" in subsequent issues.

For "speed" problems the procedure is quite different. Often whimsical, these problems should not be taken too seriously. If the proposer submits a solution with the problem, that solution appears at the end of the same column in which the problem is published. For example, solutions to this issue's "speed" problems are given below. Only rarely are coments on "speed" problems published or acknowledged.

There is also an annual problem, published in the first issue of each new year; and sometimes I go back into history to republish problems which remained unsolved after their first appearance.



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All problems come from readers, and all readers are invited to submit their favorites. I'll report on the size of the backlog, and on the criteria used in selecting solutions for publication, in a future issue.

Finally, let me correct an error from the July issue: Mr. Yoshigahara has named me "world puzzler" number *8*, not number 6 as stated in July.

Problems

OCT 1. We begin with a bridge problem from Howard Sard, who wants to know the minimum number of high-card points needed to make a contract of 7 spades against the best defense.

OCT 2. Here's one that appeared under the title "A Parable of Marbles" in the M.I.T. Physics Department student newsletter edited by Minn Chung, '80:

Once upon a time there lived a king whom everybody called the King of Marbles. He decorated every wall and floor of his palace with colorful marbles and paved the roads to the palace with gold marbles. But his obsession over marbles had an interesting history. You see, when he was just a kid, his mother, the queen, spanked him whenever he played with marbles. But after his mother had died and he had become king there was no one to stop him from playing with marbles. So his libido exploded, and one fine evening he made a declaration to his subjects: "Here I have a hollow semi-sphere with a hole at the bottom. I shall drop this marble into the sphere. If any of you can stop the marble from dropping through the hole without touching it, I shall make you a very rich person." Many clever people tried various ingenious techniques such as looking at the marbles standing upside down, attempting to build an antigravity machine, and praying to whomever would listen. Obviously, none of them worked—until it was Mitsy's turn. Mitsy was not only clever but she was very wise—she knew physics from 8.01, you see. She built a machine with which she could rotate the sphere, and when the king dropped the marble into the rotating sphere, the marble stopped rolling exactly halfway from the bottom. So Mitsy became very rich and famous (the king liked her so much he wanted her to be queen), and everybody lived happily ever after.

Question: Assuming the frictional coefficient between marble and sphere is μ and the radius of the sphere is R, find the maximum angular velocity of Mitsy's sphere.

OCT 3. Jonathan Hardis offers us a problem from the realm of applied probability or theoretical gambling:

In the Illinois Lottery Lotto game, the player chooses six different integers from 1 to 40. If the six match, in any order, the six different integers drawn by the lottery, the player wins the grand prize jackpot which starts at $1 million and grows weekly until won. Multiple winners split the pot equally. For each $1 bet, the player must pick two, presumably different, sets of six integers. Considering the grand prize alone, under what conditions would it pay, on the average, to play this game? In the game week ending June 18, 1983, 78 people matched all six winning integers and split the jackpot. Estimate the odds of this outcome, given that 2 million people bought $1 tickets that week.

OCT 4. Mearle Smith has a problem involving an "associate research professor" (I wonder where she got the funny title and why it is that I don't like the solution to the problem):

An associate research professor walks into his office one morning and says to his secretary, "I had three dinner guests last night. The product of their ages was 2450. The sum of their ages was twice your age. Can you tell me their three ages?" Ten minutes later his secretary came to him and said that she could not solve the problem. He said, "You are right. I will now tell you that I was the oldest one there." She was then able to tell him the ages of the three dinner guests. What are the ages of the three dinner guests, her age, and the professors's age?

OCT 5. Irving Hopkins asks: Given an irregular polygon of n sides, in which sequence should the sides be arranged and how should the corner angles be determined to give the greatest area?

Speed Department

SD 1. Bruce Calder writes:

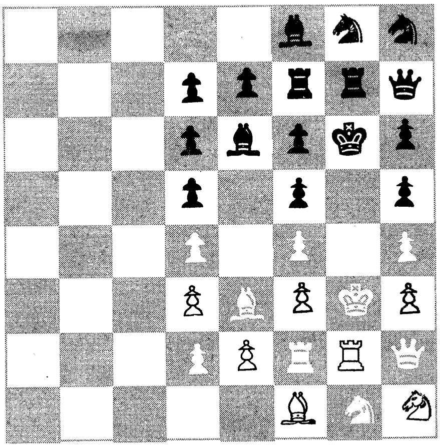
A telegram attached to a crate containing six canisters of 1000 pellets each reads: "Crate may contain canisters of defective pellets. Defects weigh 1 milligram (+/− 1 nanogram) less than proper 1 gram (+/− 1 microgram)." Using a scale accurate to 100 micrograms, what is the minimum number of weighings required to identify the defective canisters?

SD 2. What's wrong with Lester Steffen's "proof" that any number x must equal 2. Recall from alegebra I that $a^{b^c} = a^{bc}$. If we let a = b = c, we get $x^{x^x} = x^{xx} = x^{x^2}$. Thus x must be 2.

Solutions

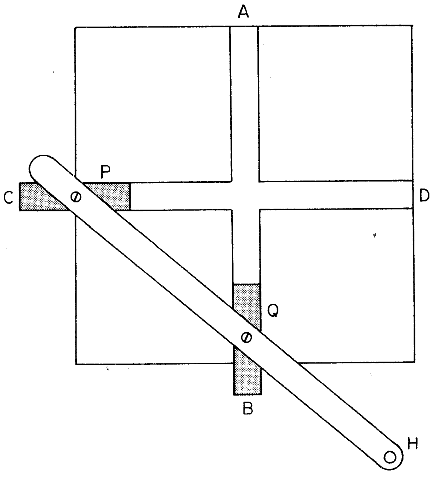
M/J 1. Given all 32 pieces, place them on the chess board in such a way that neither player can move or capture a piece. Pawns can be on any rank and doubled, tripled, even quadrupled upon the same file. No piece can be considered a promoted pawn (use of three bishops and only seven pawns is illegal).

Although pawns were allowed on any rank, I have chosen Richard Hess's solution primarily because he alone was able to avoid the normally illegal first and last ranks:

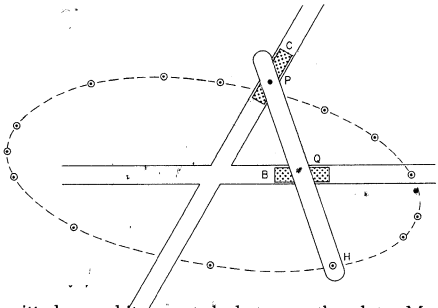


Also solved by Everett Leroy, Samuel Levitin, Matthew Fountain, Roger Spellman, and the proposer, John Walz.

M/J 2. Analysis of the device shown below proves that the orbit of H is an ellipse as H is rotated to a full 360° with travellers B and C moving in their respective slots. If the two slots are at 60° instead of 90° the figure similarly drawn (*top of page A31*) seems to be an ellipse, and if the axes are of the same length the two ellipses seem to coincide. Prove whether the 60° configuration is a true ellipse. Can the tilt of the major axis be calculated?

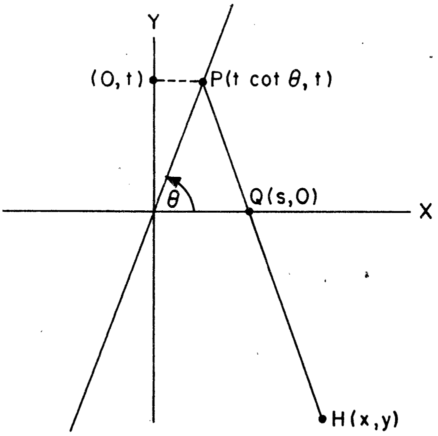


Preventing a further generalization of this problem in 1985 or 1986, Charles Sutton and others per-



mitted an arbitrary angle between the slots. Mr. Sutton writes:

Considering a more general problem, let's assume one slot is inclined at an angle θ to the other slot. Choosing coordinates as shown in the diagram, let the projection of P on the y-axis be (O,t) and the coordinates of Q be (s,O). Then the coordinates of P will be ($t \cot \theta, t$), and we can let the coordinates of H, the equation of whose path we wish to find, be (x,y).



Letting the fixed distance from P to Q be a, we have

$(t \cot \theta - s)^2 + t^2 = a^2$ (1)

Also, letting the ratio of the lengths of PH to PQ be r and expressing the condition that P, Q, and H are colinear, we have

$x - t \cot \theta = r(s - t \cot \theta)$ (2)

$y - t = r(0 - t).$ (3)

Multiplying (3) by $\cot \theta$ and subtracting from (2) gives $x - y \cot \theta = rs$, which when solved for s gives

$s = (x - y \cot \theta)/r.$ (4)

Solving (3) for t gives

$t = y/(1 - r).$ (5)

Substituting the values of s and t from (4) and (5) into (1) and simplifying we obtain the cartesian equation of the curve:

$(r - 1)^2x^2 + 2(r - 1)(\cot \theta)xy + (r^2 + \cot^2\theta)y^2 - a^2r^2(r - 1)^2 = 0$ (6)

Comparing this with the standard form of the second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, we find that the discriminant $B^2 - 4AC = 4(r - 1)^2\cot^2\theta - 4(r - 1)^2(r^2 + \cot^2\theta) = -4r^2(r - 1)^2\cot^2\theta$, which is clearly negative, showing that (6) represents an ellipse. Again, the angle α through which the coordinate axes must be rotated to remove the xy term from equation (6), and hence make the axes of the ellipse coincide with the coordinate axes, is given by

$\cot 2\alpha = (A - C)/B = (1 - 2r - \cot^2\theta)/[2(r - 1)\cot \theta]$, (7)

which depends on both r and θ. When $\theta = 60°$, $\cot \theta = \sqrt{3}/3$ and (7) becomes

$\cot 2\alpha = [(1 - 3r)\sqrt{3}]/3(r - 1),$ (8)

from which the angle α can be calculated. For example, when PQ = QH, so r = 2, $\cot 2\alpha = 5\sqrt{3}/3 = -2.88675$, so $2\alpha = -19.107°$ and $\alpha = -9.553°$; so the major axis of the ellipse will be inclined at this angle to the horizontal slot (the x-axis). If r becomes very large, $\cot 2\alpha$ will be nearly $-\sqrt{3}$ so approximately $2\alpha = -30°$ and $\alpha = -15°$. Note also that r can be less than one, implying that point H is between P and Q. Thus if r = 1/3, $\cot 2\alpha = 0$ so $2\alpha = 90°$ and $\alpha = 45°$.

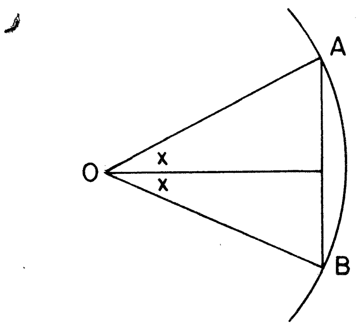
Also solved by Karl Brendel, John Langhaar, Matthew Fountain, Winslow Hartford, Harry Zaremba, William Peirce, and Richard Hess.

M/J 3. Find the Maclaurin series for sin and cos *without* using calculus or higher mathematics, although the concept of limits may be used. By way of an analogy, $(1 + x/n)^n$ can be expanded using the bionomial theorem for n a positive integer (hence derivable without calculus) to give, after simplifying

$1 + x + (1 - 1/n)\, x^2/2! + (1 - 1/n)(1 - 2/n)x^3/3! + \ldots,$

which, upon letting x approach infinity, becomes the Maclaurin series for e^x.

Several readers noted that sin and cos can be expressed in terms of the exponential via complex analysis. I am printing two proofs using more elementary approaches, one geometric and the other based on DeMoivre's theorem (which can be proved using just the double angle formula and the definition of i). First we have from Matthew Fountain:



In the figure AB is a chord on a circle of unit radius. The area of triangle OAB is both (1/2) sin 2x and (sin x)(cos x). Thus,

$(1/2)\sin 2x = (\sin x)(\cos x).$ (1)

From this equation is derived

$\cos 2x = 2\cos^2 x - 1.$ (2)

As chord AB on the unit circle decreases in length, its length approaches that of arc AB, sin x approaches x, and cos x approaches $(1 - x^2)^{1/2}$ or, equivalently, $1 - (1/2)x^2$. This shows that the power series for cos x must begin

$\cos x = 1 - (1/2)^2x^2 + \ldots$

Also, as $\cos x = \cos -x$, the power series contains only even powers of x. Setting

$\cos x = 1 - (1/2)x^2 + ax^4 + bx^6 + \ldots$

in equation (2) results in

$1 - (1/2)(2x)^2 + a(2x)^4 + b(2x)^6 + \ldots =$
$2[1 - (1/2)x^2 + ax^4 + bx^6 + \ldots]^2 - 1.$

Equating coefficients of terms of similar power,

$16a = 4a + (1/2),$
$64b = 4b - 2a, \ldots$

Solving, a = (1/24), b = (−1/720), . . . This gives the Maclaurin series,

$\cos x = 1 - (1/2!)x^2 + (1/4!)x^4 - (1/6!)x^6 + \ldots$

The power series for sin x contains only odd powers of x as $\sin x = -\sin -x$. Setting

$\sin x = x + rx^3 + sx^5 + tx^7 + \ldots$

and substituting in equation (1),

$(1/2)[2x + r(2x)^3 + s(2x)^5 + t(2x)^7 + \ldots]$
$= [x + rx^3 + sx^5 + tx^7 + \ldots][1 - (1/2!)x^2 + (1/4!)x^4 - (1/6!)x^6 + \ldots].$

Equating coefficients of terms of similar power,

$4r = -(1/2) + r,$
$16s = (1/4!) - (1/2)r + s,$
$64! = -(1/6!) + (1/4!)r - (1/2)s + t. \ldots$

Solving, r = −(1/3!), s = (1/5!), t = −(1/7!), . . . This gives the Maclaurin series,

$\sin x = x - (1/3!)x^3 + (1/5!)x^5 - (1/7!)x^7 + \ldots$

The second solution is from Harry Zaremba: The series can be determined by using DeMoivre's theorem,

$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n.$

If the right side is expanded and the real parts of the left and right sides are equated, we have

$\cos n\theta = \cos^n\theta - [n(n - 1)\cos^{n-2}\theta \sin^2\theta]/2! + [n(n - 1)(n - 2)(n - 3)\cos^{n-4}\theta \sin^4\theta]/4! + \ldots$

Let $x = n\theta$. Then $\theta = x/n$ and $n = x/\theta$ in which x is to remain constant while n and θ vary. Substituting the values into (1),

$$\cos x = \cos^n\theta - \frac{\frac{x}{\theta}\left(\frac{x}{\theta} - 1\right)}{2!}\cos^{n-2}\theta\sin^2\theta + \frac{\frac{x}{\theta}\left(\frac{x}{\theta} - 1\right)\left(\frac{x}{\theta} - 2\right)\left(\frac{x}{\theta} - 3\right)}{4!}\cos^{n-4}\theta\sin^4\theta + \dots$$

$$\cos x = \cos^n\theta - \frac{x(x-\theta)}{2!}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \frac{x(x-\theta)(x-2\theta)(x-3\theta)}{4!}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4 + \dots$$

If $n \to \alpha$, then $\theta \to 0$, $\cos\theta \to 1$, $\sin\theta/\theta \to 1$, and $(x - \theta) \to x$. Therefore
$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \dots$
If the imaginary part of the left side is equated to the imaginary parts of the expanded right side, and similar limits are applied, the result will be
$\sin x = x - x^3/3! + x^5/5! - x^7/7! + \dots$

Also solved by John Prussing, Charles Sutton, and Richard Hess.

M/J 4. Consider the number $8888^{8888} \equiv A$. Suppose you were able to write it in decimal form. Then suppose you add its decimal digits together and therefore construct their sum $\equiv B$. Again, write B in decimal form and add its decimal digits to form a new number C. Repeat this step once more (add the decimal digits of C to form a new number). What is this last number?

The following solution is from Douglas Ell:
My first step was to obtain a rough estimate of the magnitude of the solution, which I labeled "D." The number 8888^{8888} has no more than 4×8888 digits and thus
$B \leq 4 \times 8888 \times 9 < 320{,}400$.
Continuing,
$C \leq 2 + 5 \times 9$; $D \leq 13$.
Let R be the function on the positive integers obtained by continuing the process described above (converting A to B to C to D) until a single-digit number is obtained. Note that this process consists of subtracting various multiples of 9. Thus $R(m) = n$, where $m \equiv n(\text{mod}9)$ and $1 \leq n \leq 9$. Also, if i and j are positive integers, $R(i \times j) = R(R(i) \times R(j))$ and thus $R(i^j) = R[(R(i))^j]$. If follows that $R[8888^{8888}] = R[5^{8888}] = R[R(5^2) \times R(5^{6\times1481})] = R[R(25) \times 1] = 7$. (Note that $5^6 \equiv 1(\text{mod}9)$.) Because D is at least a partial step in obtaining $R(8888^{8888})$, it follows that $D \equiv R(8888^{8888})(\text{mod}9)$.
Putting together both $1 \leq D \leq 13$ and $D \equiv 7(\text{mod}9)$, we see that $D = 7$.

Also solved by Karl Brendel, John Langhaar, Harry Lieberman, Steve Fieldman, Frank Carbin, David Simen, Matthew Fountain, Harry Zaremba, Richard Hess, and the proposer, Anthony Beris.

M/J 5. "None of my children is over twenty," said Mr. Euclid, "and I notice that this year the sum of the cubes of the ages in years of the younger three is equal to the cube of the age of the eldest." "I can't tell their ages from that information," said his friend Mr. Diophant, "even though I know the age of the eldest of your four kids." What was the age of Mr. Euclid's third child?

I find it fitting that Caryln Iuzzolino let Pascal help out Euclid and Diophant:
The program I wrote is

```
program ijkl (input,output);
var i,j,k,l:integer;
begin
for i: = 1 to 20 do
for j: = 1 to i do
for k: = 1 to j do
for l: = 1 to k do
if i*i*i=j*j*j+k*k*k+l*l*l
then writeln(i,j,k,l);
end.
```

The output:

6	5	4	3
9	8	6	1
12	10	8	6
18	15	12	9
18	16	12	2
19	18	10	3
20	17	14	7

Since the friend couldn't tell the ages, even though he knew how old the oldest child was, the oldest child must have been 18 and therefore the third oldest was 12 years old.

Also solved by Karl Brendel, Robert Slater, Richard Hess, Matthew Fountain, Harry Zaremba, Evan Klein, Edward Amrein, Avi Ornstein, Yale Zussman, Mel Garelick, Clarence Cantor, Phelps Meaker, Dennis Sandow, Winslow Hartford, John Prussing, Steve Feldman, Frank Carbin, David Simen, Samuel Levitin, Frank Conlin, Gardner Perry, Richard Marks, and the proposer, John Hughes.

Better Late Than Never

1982 OCT 2. The proposer (who has had trouble getting his *Review* in Japan) reports that he "was astonished to see" the solution given in the March 1983 issue. He enclosed the following solution and asks us to pay particular attention to the *n given in the sums.

```
    13744
    64948
    64948
+   64948
   104294 *2

      645
    75054
    75054
    75054
    75054
+   75054
   125305 *3

    84502
    84502
    68387
    68387
    68387
+  818387
   298138 *4

     5394
     5394
    24947
    24947
    24947
    24947
   404947
+  404947
   184094 *5
```

1983 JAN 1. The proposer Doug Van Patten responds to Ruth by Turner's query. North should have been declarer, not South. North started the bidding with a club and East-West competed. After bidding clubs twice, North tried a 4-spade bid, which his partner accepted. So the bidding was natural—no contorted systems! The position of the declarer does not affect the solution.

F/M 4. Phelps Meaker has responded.

APR 4. Howard Stern has responded.

M/J SD1. J. Lawrence and H. Shaw report that the answer is 5.

Proposer's Solutions to Speed Problems

SD 1. One weighing suffices. Mark the canisters 1, 2, 4, 8, 16, and 32, and remove the respective number of pellets from each. The number of milligrams less than 63 grams uniquely identifies the defective canisters. For example, if scale reads 62.9730765 grams, then 27 milligrams shy. Therefore 27 = 16 + 8 + 2 + 1 so the canisters marked 1, 2, 8, and 16 are defective.

SD 2. $x^{x^x} = x^{(x^2)} = x2^{x^2}$