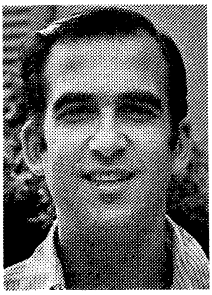
How to Open a Padlock

Puzzle Corner/Allan Gottlieb



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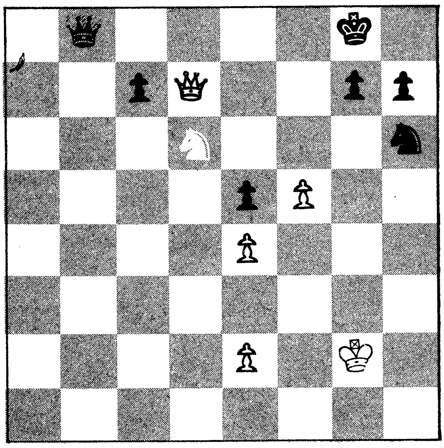
Joseph Horton suggests that I run a new class of puzzle—computer-oriented challenges in which the goal is to compose and run a program to solve a given problem. Mr. Horton further suggests that I specify a particular language and judge solutions on execution time. Since the latter quantity is very machine-dependent, I will permit a class of languages and use the same previously-described subjective criteria to select the solution for publication. To ease typesetting problems, only standard (ascii) characters are permitted; and for readability assembly language is forbidden. I am sorry that APL is ruled out but believe it is necessary. Languages permitted would include Ada, Basic, C, Fortan, Pascal, PL/I, etc. A possible problem in this class would be **APR 2** (*see below*).

As soon as computer-related problems start arriving, I will intersperse them with the monthly chess and bridge offerings. Reader interest will determine their fate. So if you have any such problems, send them in now; the backlog can never be less than it is currently.

Problems

A/S 1. We begin this month with a chess problem (*see next column*) from Bob Kimble (who credits Robert Brieger's *Imagination in the End Game*) that bears a familiar request: "White to move and win."



A/S 2. The following problem is from John Prussing:
An ordinary combination padlock requires three ordered numbers to open, each between 0 and 39, inclusive. Thus there are 64,000 possible combinations. If it is known that the sum of the three numbers is 58 and the sum of the individual digits of all three numbers is 13, how many combinations are possible? If each of these possible combinations is equally likely, what is the probability that the second number is 34?

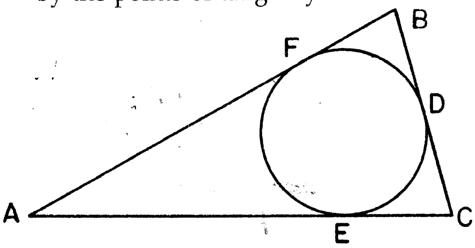
A/S 3. Phelps Meaker has looked at the regular polyhedra and produced the table below. He would like you to fill in the missing entry and wonders if the values in the last column can be determined by a formula. [Of course, there is a fourth-degree polynomial relating these five values to the number of sides, but that is not what is wanted.—*Ed.]*

A/S 4. Matthew Fountain wants you to find infinitely many positive integers n not containing the digit zero such that $n^2 - 1$ contains just two digits neither of which is zero. Note that we are not requiring that $n^2 - 1$ is a two-digit num-

Number of sides	Edges each side	Total edges	Number of dihedrals	Dihedrals meeting at each apex	Number of apices
4	3	12	6	3	4
6	4	24	12	~~4~~ 3	~~6~~ 8
8	3	24	12	4	6
12	5	60	30	3	20
20	3	60	30	5	?

ber—i.e., the digits may be repeated.

A/S 5. Our last problem is from Harry Zaremba:
Given a triangle ABC, draw its incircle and consider triangle DEF determined by the points of tangency.



Show that the area of DEF is (r/d)A, where r is the radius of the incircle, d is the diameter of the circumcircle, and A is the area of ABC.

Speed Department

SD 1. Irving Hopkins wants you to find two functions F and G such that F(n!) = O and G(n!) = 1 for all $n \geq 6$.

SD 2. Phelps Meaker asks:
A rectangular pan with perpendicular sides is to be bent up from a sheet of metal 8″ by 10″. Forgetting about tight seams, what is the depth of the pan if it holds 52.5 cu. in.?

Solutions

APR 1. West is defending a six-no-trump contract and has led the ♣A. How can West meet this contract?

	♠ Q 10 3 ♥ Q 7 ♦ A Q J 8 6 4 ♣ J 9	
♠ 9 8 7 ♥ 8 3 ♦ 10 9 7 5 2 ♣ A 10 6		♠ J 6 5 2 ♥ J 10 7 2 ♦— ♣ Q 8 5 4 2
	♠ A K 4 ♥ A K 9 5 4 ♦ K 3 ♣ K 7 3	

The trick is to break up the threatened squeeze. You either throw it high and tight to a right-hand hitter or lead a low club. John Bobbitt chose the latter. West can defeat the contract, but only by leading the ♣6. A diamond lead could allow South to bring home the entire diamond suit. A heart or spade lead will allow South to successfully squeeze both East and West. Assume a spade is lead (it doesn't matter which). South takes the ♠K and the three top diamonds. East discards two clubs and one spade, South discards a heart. South now takes the three top hearts, and the ♠A. West discards a spade on the third heart, with all other leads following suit. This leads to the following situation:

	♠ Q ♥— ♦ J 8 ♣ J	
♠— ♥— ♦ 10 7 ♣ 10 6		♠ J ♥ J ♦— ♣ Q 8
	♠ 4 ♥ 9 ♦— ♣ K 7	

South leads the ♠4. West cannot part with a diamond, or else South will get the ♦J and ♦8 and the ♣K. So West discards a club (the first squeeze). Now dummy leads the ♦J, and East is squeezed. If East discards a heart, South discards the ♣7, and the ♣J is lead to the ♣K and ♥9. If East discards a club, South discards a heart, and South's two clubs are good. Note how important it is for South to retain the ♣K and ♣7 until the end. By leading the ♣6 at trick 2, West deprives South of this double squeeze. Furthermore, there is no other combination South can use to squeeze East-West out of at least one more trick.

Also solved by Woody Pidcock, Robert Bart, Winslow Hartford, Matthew Fountain, Philip Dangel, Alan Berger, and the proposer, Doug Van Patter.

APR 2. In this cryptarithmetic problem—a multiplication problem involving time—each of the 10 digits is to be used once. The first number represents minutes and seconds; the third, hours, minutes, and seconds; and the leftmost digit of each number is nonzero:

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     x xx
       xx
  -------
  x xx xx
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Apparently there are four solutions, as pointed out by Antony Beris:

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     3 02        6 47        4 18        5 09
       98          19          72          84
  -------     -------     -------     -------
  4 57 16     2 08 53     5 09 36     7 12 36
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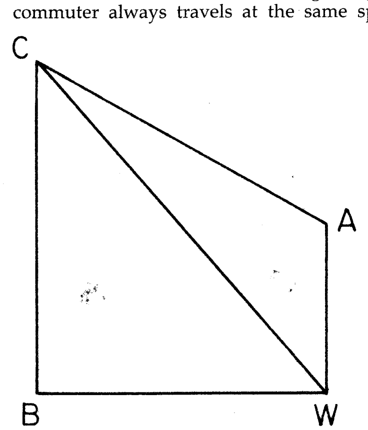
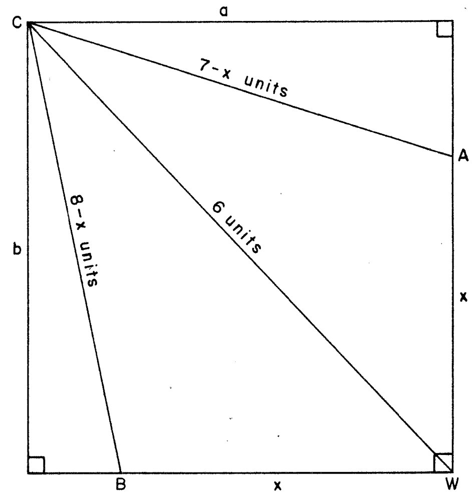
Also solved by Brian Almond, Dennis Sandow, Samuel Levitin, Matthew Fountain, Robert Bart, James Brown, Steven Feldman, and the proposer, Nob. Yoshigahara

APR 3. A commuter lives at C and works at W. He normally drives to work via road CW. Occasionally, to break the monotony of the daily commute, he drives to Town A and from there to W. On still other occasions he drives to Town B and from there to W. Towns A and B are each exactly 10 miles from W and roads BW and AW are at right angles. Our commuter always travels at the same speed, regardless of route. The normal trip takes exactly 30 minutes, the trip through Town A takes 35 minutes, and the trip through Town B takes 40 minutes. At what rate of speed does this commuter travel?



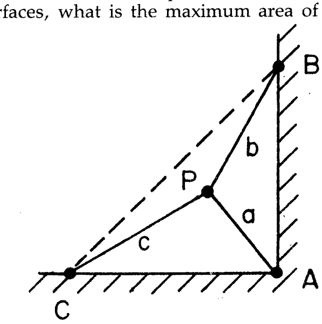
William Veeck, Peter Card, and the proposer, A. Singer, noted that his problem caused quite a flurry of activity when it appeared in *Popular Science*. The following solution is from first-time respondent Woody Pidcock:
The diagram below shows the definition of variables a, b, and x. To keep the magnitude of numbers small, I define x as the time it takes the commuter to travel 10 miles in "5-minute units." The lengths in the figure and the numbers in the following equations represent values in these "time" units.



(1) $a^2 + b^2 = 36.$
(2) $(a - x)^2 + b^2 = (8 - x)^2.$
(3) $a^2 + (b - x)^2 = (7 - x)^2$
Adding (2) and (3), and subtracting (3) from (2),
(4a) $(30 - 2a - 2b)x = 41.$
(4b) $b - a = (30x - 41)/2x.$
(5a) $(2 - 2b - 2a)x = 15.$
(5b) $b - a = (-2x + 15)/2x.$
Adding (4b) and (5b) to solve for b, and subtracting (5b) from (4b) to solve for a,
(6) $a = (16x - 28)/2x.$
(7) $b = (14x - 13)/2x.$
Substituting (6) and (7) into (1) and solving by quadratic formula,
(8) $308x^2 - 1260x + 953 = 0.$
(9a) $x_1 = (1260 + 643.043)/616 = 3.089.$
(9b) $x_2 = (1260 - 643.043)/616 = 1.002.$
From the definition of x above, (9a) results in a rate of speed of 10 miles/(3.089 × 5 minutes) = 38.85 miles/hour. (9b) results in a speed of 10/(1.002 × 5) × 60 = 119.76 miles/hour, which is excessive for daily commuting and quite foolish. Town B would be on the opposite side of town W in a direct commute from town C.

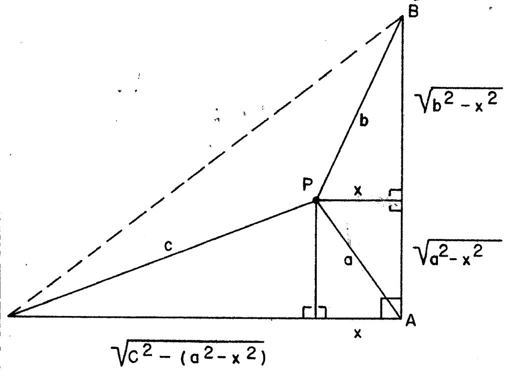
Also solved by Leo Harten (and MACSYMA), Brian Almond, Phelps Meaker, Frank Carbin, Winslow Hartford, Matthew Fountain, Richard Gould, Rhea Graham, Avi Ornstein, John Bobbitt, Norman Wickstrand, Mary Lindenberg, A.C. Lawson, Harry Garber, P.V. Heftler, and the proposer.

APR 4. In the figure, rigid rods of lengths a, b, and c are hinged at point P, and rod a is hinged to the intersection of the vertical and horizontal surfaces. If the free ends of the rods b and c are always maintained in contact at B and C and permitted to slide along the surfaces, what is the maximum area of



the right triangle CAB that can be formed by the rod extremities? (For uniformity and simplicity, let the constant $K^2 = b^2 + c^2 - a^2$.)

The following is from Harry Garber: Begin by extending perpendiculars from P to the legs of a right triangle CAB, and let x equal the distance from P to AB. Then the Pythagorean theorem provides the lengths indicated in the sketch.



Then the leg lengths and desired areas for a given x are: $AB = (a^2 - x^2)^{1/2} + (b^2 - x^2)^{1/2}$, $AC = x + (c^2 - a^2 + x^2)^{1/2}$. Area = 1/2(AB)(AC).

To maximize the area, find the x which gives $[d(Area)]/dx = 0$. The task is more tractable, I believe, if rather than multiplying the area out and then differentiating, we use the product rule of derivatives:

$$\frac{d(Area)}{dx} = \frac{1}{2}\left[(\overline{AB})\frac{d(\overline{AC})}{dx} + (AC)\frac{d(\overline{AB})}{dx}\right].$$

Since

$$\frac{d(AB)}{dx} = -\frac{x}{(a^2 - x^2)^{1/2}} - \frac{x}{(b^2 - x^2)^{1/2}}$$

and

$$\frac{d(AC)}{dx} = 1 + \frac{x}{(c^2 - a^2 + x^2)^{1/2}},$$

one just multiplies out the result, clearing each term to a separate common denominator, and simplifies to obtain

$$0 = \frac{1}{2}\left[(\overline{AB})(\overline{AC})\frac{1}{(c^2 - a^2 + x^2)^{1/2}} - \frac{x}{(a^2 - x^2)^{1/2}(b^2 - x^2)^{1/2}}\right]$$

Since the area is

$$\frac{1}{2}(\overline{AB}), (\overline{AC}),$$

$$0 = (Area)\left[\frac{1}{(c^2 - a^2 + x^2)^{1/2}} - \frac{x}{(a^2 - x^2)^{1/2}(b^2 - x^2)^{1/2}}\right]$$

and we seek to maximize the area, not minimize it by having it be zero. Thus the rightmost factor is zero, leading to

$$x = \frac{ab}{(b^2 + c^2)}.$$

Plenty of algebra (which I'll omit) follows upon substituting the x back in; I find

$$(\overline{AB}) = \frac{1}{(b^2 + c^2)^{1/2}}(ac + bK),$$

$$(\overline{AC}) = \frac{1}{(b^2 + c^2)^{1/2}}(ab + cK).$$

$$\text{Area} = \frac{1}{2}[bc + aK].$$

Also solved by Harry Garber, Mary Lindenberg, Ben Abunar, A.C. Lawson, Matthew Fountain, Woody Pidcock, and the proposer.

APR 5. The squares of an infinite chessboard are numbered by putting zero in the corner and in each other square putting the smallest non-negative integer that does not appear to its left in the same row or below it in the same column. Find a non-recursive formula for the number placed in row i and column j.

Many readers tripped up here and obtained the incorrect formula i + j − 2. For a two-by-two board this gives

1 2
0 1

instead of

1 0
0 1

However, Matthew Fountain was not fooled and writes:

The number = (i − 1) XOR (j − 1). The XOR operator compares the corresponding digits of two binary numbers and returns a number which contains the digit 1 in those positions where a match is not found and a O in those positions where a match is found. The relationship was found during my construction of the lower left corner of the chessboard. I began by placing the O's, starting with the first row, then the second row, and so on, and noticed that they made a repeating, one-digit pattern up the long diagonal of the chessboard. When I placed the 1's, I noticed the pattern in the first two rows and first two columns was repeated up the long diagonal. After I placed the 2's and 3's, I noticed the pattern in the first four rows and the first four columns repeated up the long diagonal:

7	6	5	4	3	2	1	0
6	7	4	5	2	3	0	1
5	4	7	6	1	0	3	2
4	5	6	7	0	1	2	3
3	2	1	0	7	6	5	4
2	3	0	1	6	7	4	5
1	0	3	2	5	4	7	6
0	1	2	3	4	5	6	7

This suggested renumbering the squares in binary notation.

11	10	01	00
10	11	00	01
01	00	11	10
00	01	10	11

Then it was apparent that every binary number in the square was equal to the result of applying the XOR operation on the number furthest to its left in the same row and furthest below it in the same column. These numbers in turn are equal to one less than the column or row that they are in. Therefore the number at row i, column j is (i − 1) XOR (j − 1).

Also solved by Brian Almond, P.V. Heftler, Woody Pidcock, Robert Bart, and Harry Garber.

Better Late Than Never

NS 2. Angus Lawson sent us a reprint on this problem from the October 1982 issue of the *American Journal of Physics*. Mr. Lawson offers the following explanation for his 1984 response to our 1976 problem:

I worked out the enclosed solution to **NS2** at the same time your excellent column disppeared from my edition of *Technology Review*. Fearing the worst (and being too lazy to check), I published elsewhere, giving you proper credit. Now that I've become a fake alumnus, I have rejoined the ranks of your readers and hope you will enjoy the reprint.

1983 M/J 4. William Veeck found an alternative solution.

N/D 1. Robert Bart found that R−f4 also leads to mate in 3.

1984 F/M 1. Elliott Roberts has responded.

F/M 2. Michael Jung and T. Landale have responded.

F/M 4. Michael Jung has responded.

APR SD1. Mary Lindenberg and Woody Pidcock noticed that the sequence is n^{π}.

Proposers' Solutions to Speed Problems

SD 1. sin and cos.

SD 2. 1.5 inches.