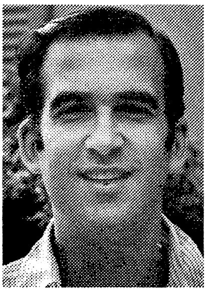
How Old Are Euclid's Children?

Puzzle Corner/Allan Gottlieb



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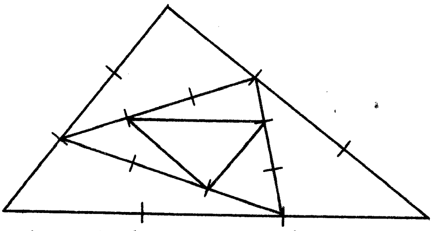
Daniel Shapiro's article in the February *American Mathematical Monthly* contains a result that I think you will find interesting.

It is easy to see that if the midpoints of the three sides of a triangle are connected, the resulting triangle is similar to the original one. If you use trisection

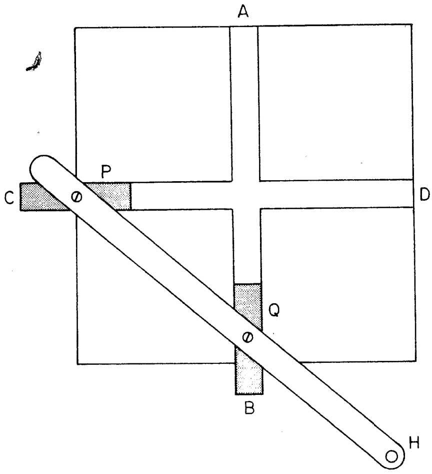


points, similarity occurs after two iterations, i.e., the smallest triangle in the figure above is similar to the largest. A natural question to ask is what happens for ratios like 1/5, 2/7, or other rationals? Shapiro shows (among other things) that 1/2 and 1/3 are special; for all other rationals the triangles determined by successive iterations are all dissimilar (unless the original triangle is regular or degenerate). I was surprised; are you?

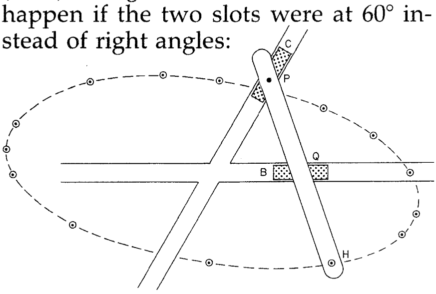
Problems

M/J 1. We begin with a chess problem from John Walz, who adds that until recently he believed that no solution was possible:
Given all 32 pieces, place them on the board in such a way that neither player can move or capture a piece. Pawns can be on any rank and doubled, tripled, even quadrupled upon the same file. No piece can be considered a promoted pawn (use of three bishops and only seven pawns is illegal).

M/J 2. Phelps Meaker enjoyed **1983 JUL 4** and now proposes a variant. The original problem called for the analysis of the device shown at the top of the next column, proving that the orbit of H is an ellipse as H is rotated a full 360° with travellers B and C moving in their respective slots.
Now Mr. Meaker writes:



After the fun with the trammel of JUL 4 (1983), I began to wonder what would happen if the two slots were at 60° instead of right angles:



The figure seems to be an ellipse. I drew an ellipse with axes of the same length with two pins and a string, and they seem to coincide. Prove whether it is a true ellipse. Can the tilt of the major axis be calculated?

M/J 3. Reino Hakala wants you to find the Maclaurin series for sin and cos *without* using calculus or higher mathematics, although the concept of limits may be used. By way of an analogy, $(1 + x/n)^n$ can be expanded using the binomial theorem for n a positive integer (hence derivable without calculus) to give, after simplifying
$1 + x + (1 - 1/n)\, x^2/2! + (1 - 1/n)(1 - 2/n)\, x^3/3! + \ldots,$
which, upon letting x approach infinity, becomes the Maclaurin series for e^x.

M/J 4. Anthony Beris recalls a problem he encountered at a math Olympiad: Consider the number $8888^{8888} \equiv A$. Suppose you were able to write it in decimal form. Then suppose you add its decimal digits together and therefore construct their sum $\equiv B$. Again, write B in decimal form and add its decimal digits to form a new number C. Repeat this step once more (add the decimal digits of C to form a new number). What is this last number?

M/J 5. Our final regular problem is from John Hughes:
"None of my children is over twenty," said Mr. Euclid, "and I notice that this year the sum of the cubes of the ages in years of the younger three is equal to the cube of the age of the eldest." "I can't tell their ages from that information," said his friend Mr. Diophant, "even though I know the age of the eldest of your four kids." What was the age of Mr. Euclid's third child?

Speed Department

SD 1. Here's one from Phelps Meaker: An open-top cubical tank has an internal area in square feet equal to the number of cubic feet of water it can hold. How big is the tank?

SD 2. Steven Kanter wants you to determine the hands and the grand slam bid so that no matter who the declarer is, the defenders get all of the tricks!

Solutions

JAN 1. South is declarer at the contract of four spades:

	♠ J 9 8 5 2 ♥ A ♦ – ♣ A K Q 10 9 6 4	
♠ A Q 3 ♥ K J 3 ♦ A J 10 8 6 ♣ J 2		♠ K 7 ♥ 9 6 5 4 2 ♦ K Q ♣ 5 3
	♠ 10 6 4 ♥ Q 10 8 7 ♦ 5 4 3 2 ♣ 8 7	

West leads the ♦A. Would you choose to be declarer or defender?

Smith D. Turner (ʃdt) gave the missing ♦7 and ♦9 to East, notes that Ruth Turner wonders what contorted bidding system could result in a spade contract, and writes:
The defense can be held to the three top trump tricks. Defense is on the right track in opening and continuing diamonds to reduce North's trumps and reentries, but this line will not reach far enough. The most difficult defense I found to overcome is:

	West	North	East	South
1.	♦A	♠2	♦7	♥2
2.		♠5	♠7	♠10
	♠A			
3.	♦6	♠8	♦9	♦3
4.		♠9	♠K	♠4
	♠3			
5.			♦K	♦4
	♦J	♠J		
6.		♣A	♣J	♣7
	♣2			
7.		♣K	♣3	♣8
	♣J			
8.		♣Q	♥2	♦5

If West does not trump at trick 8, North continues clubs until West does trump. After trumping, if West leads a heart the board is good; if West leads a diamond, South trumps, leads a heart, and the board is good. It does not help the defense to knock out this last heart reentry by leading a heart on, say, trick 3, as this saves North from using a spade for reentry.

Also solved by Alan Robock, Matthew Fountain, Peter Silverberg, Robert Bart, Warren Mathews, Winslow Hartford, Richard Hess, and the proposer, Doug Van Patter.

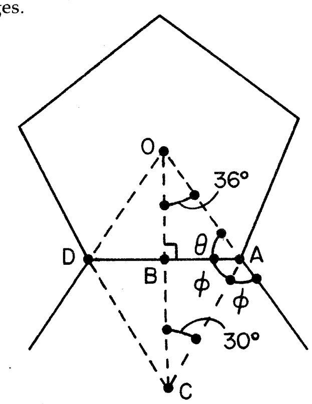
JAN 2. It is found that 91.3 percent of *Technology Review* readers enjoy "Puzzle Corner," a value accurate to three digits. What is the minimum number of readers the reporter could have polled for this value to be so accurate?

Robert Bart, unlike the editor, prefers to consider respondents who dislike "Puzzle Corner." If .913 enjoyed "Puzzle Corner," .087 did not. N being the number of respondents, and since responses are assumed to be integers, .087N = i where .087 is correct to three places. For i = 1, N cannot be an integer but for i = 2, N = 23 is very close to 2/.087. This is therefore the smallest survey sample with 21 of 23 favorable responses producing the result.

Also solved by Winslow Hartford, Steven Feldman, Robert Bart, Peter Silverberg, Phelps Meaker, Frank Carbin, David Gluss, Matthew Fountain, John Prussing, Harry Zaremba, R. Raines, Richard Hess, Avi Ornstein, and the proposer, Greg Huber.

JAN 3. The soccer ball in current use appears to be made from 32 pieces of leather, 12 black pieces in the shape of regular pentagons and 20 white pieces in the shape of regular hexagons. Instead of forming a polyhedron, air pressure pushes the sides out into a circumscribing sphere. How many vertices does the basic polyhedron have? Do all of these vertices lie on the circumscribing sphere? If the edge of each pentagon and each hexagon is exactly 2 inches in length, calculate the diameter of the circumscribing sphere.

The following solution is from Harry Zaremba: The polyhedron that contains the given number of regular polygons is the truncated icosahedron. Its surface is composed of 12 identical patterns each of which consists of a regular pentagon encircled by five hexagons. The polyhedron has 60 vertices and 90 edges.



To obtain the ball diameter, assume that after inflation of the ball each edge of the polygons lies on a great circle and that its arc length equals 2 inches. In the figure shown, points C and O are on the sphere's surface and are equidistant from their polygon's vertices. Since the angles between the great circles passing though point O and the pentagon's vertices are equal, we have angle BOA equal to 360/(2 × 5) = 36°. Similarly, at point C of the adjacent hexagon, angle BCA equals 360/(2 × 6) = 30°. Let β be the angle subtended by arc length AB at the ball's center, and let R equal the radius of the ball. Since AB = 1″, we have,

$$R = 1/\beta \quad (1)$$

From spherical triangles BOA and BCA,

$$\cos 36° = \cos\beta \sin\theta \quad (2)$$

$$\cos 30° = \cos\beta \sin\phi \quad (3)$$

and from vertex A,

$$\theta = 180° - 2\phi \quad (4)$$

The solution of equations (2),(3), and (4) yields $\theta = 55.69064°$, $\phi = 62.15468°$; and $\beta = 11.64072° = 0.20317$ radians.

Thus, from equation (1), R = 1/0.20317 = 4.922″. Hence, the ball diameter is *D = 2R = 9.844 inches.*

The spherical excess in triangle DOA is

$$E_p = 72° + 2\theta - 180° = 2\theta - 108°$$

and the spherical excess in triangle DCA is

$$E_h = 60° + 2\phi - 180° = 2\phi - 120°.$$

The area of all pentagons will be

$$A_p = 12 \times 5 \times \pi R^2/180 \times E_p = \pi R^2/3 \times (2\theta - 108) \quad (5)$$

and the area of all hexagons will be

$$A_h = 20 \times 6 \times \pi R^2/180 \times E_h = 2\pi R^2/3 \times (2\phi - 120). \quad (6)$$

Substitution of equation (4) into (5), and addition of equations (5) and (6), yields

$$A = A_p + A_h = 4\pi R^2.$$

This is the surface area of a sphere which verifies that the vertices of the ball lie on the sphere's surface.

Also solved by Matthew Fountain, Winslow Hartford, Norman Wickstrand, Richard Hess, and the proposer, Winthrop Leeds.

JAN 4. On the planet Trayshowed in a distant galaxy, an earth scientist was asked to measure the blackbody temperature of the sun. Their day/night cycle was 36 hours, so the scientist was somewhat frazzled; however, he was spared a little work when he noticed that without moving his apparatus sunlight would enter it for a full 6 minutes. As predicted, the blackbody equivalent temperature was found to be 5500 Kelvin. What kind of clothing did our scientist wear?

Yes, Trayshowed is tres chaud! Alan Robock writes:

The scientist probably wore regular clothes, as he would have to be inside a spaceship to survive the high temperatures. He could possibly be wearing a space suit, probably with a French cut. In order to calculate the surface temperture, I first note that sunlight would enter the apparatus for 6 minutes of a 36-hour day. This is 6/(36 × 60) = 1/360 of a day, implying that the diameter of the sun subtends 1° from the planet.

If r = radius of the sun, and R is the distance from the center of the sun to the center of the planet, we can set up an approximately right triangle in the diagram below such that sin 1° = (2r)/R. If s = heat flux at the surface of the sun and S = heat flux from the sun at the mean sun-planet distance (the so-called "solar constant"), then from the inverse square for radiation,

$$S = r^2 s/R^2 = [r^2(\sin 1°)^2 s]/4r^2.$$

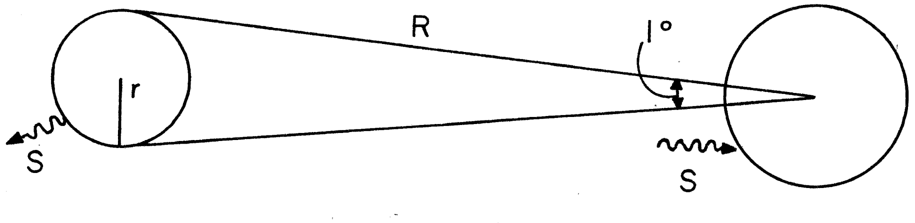
From the Stefan-Bolzmann law, we know that $s = \sigma T^4$, and T = 5500 K. We can set up an energy balance equation for the planet:

$$S(1 - \alpha)/4 = \sigma T_p^4,$$

where T_p is the planetary temperature, α is the albedo (reflectivity), and the S/4 factor comes from the 1:4 ratio of the area of the circle intersected by the sun's rays to the surface area of the planet. Substituting the above equations for S and s into the energy balance equation, we find that

$$T_p = [(\sin 1°)^2(1 - \alpha)/16]^{1/4} T.$$

The resulting surface temperature (T_s) then depends



A digits	B digits	C digits	D digits	C digits	B digits	A digits
219. . .978	0. . .0	219. . .978	0. . .0	219. . .978	0. . .0	219. . .978

on the assumptions made about the planetary albedo (α) and about the atmosphere of the planet. For the earth, $\alpha = 0.30$, and the effective radiative temperature of the planet (T_p) = 254 K = −19°C. Because the earth has an atmosphere, and some of the gases are transparent to solar radiation but absorb and re-emit in the long-wave, the so-called "greenhouse effect" is created. As a result, the average surface temperature of earth is 288 K = 15°C. For the planet in question, if there is no atmosphere (in which case $T_p = T_s$, and the above answer about the scientist's dress applies independent of temperature) and the planet is black ($\alpha = 0$), T_p = 363 K. If $\alpha = 0.5$, T_p = 305.5 K, still quite hot. If α = 0.3, as on earth, T_p = 332 K. With an atmosphere like earth's, the greenhouse effect would increase T_s to about 370 K, too hot to not be protected.

Also solved by Winslow Hartford, David Gluss, Matthew Fountain, Richard Hess, and the proposer, Dan Dewey.

Better Late Than Never

Through an error in New York compounded in Cambridge we failed in the last issue to credit Richard Hess with solutions to **N/D1, N/D2, N/D3,** and **N/D4,** and Edward Dawson with a solution to **N/D4.**

1981 OCT 3. Walter Nissen, Jr., notes that the 88-digit number consisting of the concatenation of the smallest solution to itself is also a solution, as are higher "multiples."

Y1983 Harry (Hap) Hazard notes that we need a minus sign for 56 and submits the following improvements. Have a *healthy* 1984, Dr. Hazard.

$1^{98} \times 3$
$1^{98} + 3$
$(19 + 8) \div 3$
$(91 - 3) \div 8$
$(81/3) - 9$
$1^9 + (8 \times 3)$
$(19 - 8) \times 3$
$1 \times (9 + 8) \times 3$
$1 + [(9 + 8) \times 3]$
$81 - (9 \times 3)$
$(9 - 3 + 1) \times 8$
$1 + (9 \times 8) - 3$
$(19 + 8) \times 3$
$\cdot 1^9 \times 83$
$1^9 + 83$

1983 M/J 4. William Veeck found an alternative solution.

JUL 5. Donald Savage has looked a little further and reports the following:
Nob Yoshigahara's problem led to something unexpected, as perhaps he knew. I anticipated that the next two appropriate perfect squares were not "nearby," so I had my computer do it. In short order, it came up with $(3114)^2$ = 9696996, and $(81619)^2$ = 66616661161. Then I let it continue the hunt for more numbers. After several days it had gone by 2.8×10^7 and found nothing. So I wrote a faster-running program. In about a day it had gone by 10^{11} and found nothing! So I suspect there aren't any more; but how does one prove it? (Someone should ask Gerd Faltings.) By way of clarification, my (homemade) computer looked at the square of all numbers up to the square of (10^{11}+). So how about asking your readers if they can find any more solutions?

A/S 5. Brian Almond found other solutions and writes:
The general form of X is blocks of the form 219 . . . 978 (0 or more 9's) separated by blocks of the form 0 . . . 0. X must begin and end in a block of the first form. Also, the numbers of 9's and 0's in the blocks are symmetric with respect to the center block in X. There may be an odd or an even number of blocks of the first form. For example, see the form shown across the top of this page.

OCT SD2. Lionel Bauduy found a six-step solution.

1984 JAN SD1. Warren Matthews found that the length of the walk is much closer to 959.5 inches than to 959.75 inches.

Proposer's Solution to Speed Problem

M/J SD2. The contract is 7NT and each hand has all 13 cards in a single suit.