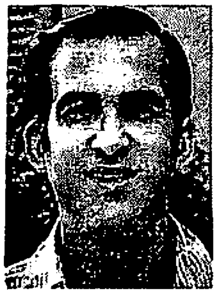
Puzzle Corner
Alan J. Gottlieb

Wanted: Quickies Quick



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Alice and I want to thank everyone for their kind words in response to our birth announcement and are pleased to add that our newly expanded family is doing just fine.

I had intended, in that same (May/June) issue, to note a critical shortage of speed problems, but somehow that portion of the introduction to "Puzzle Corner" did not appear. Thus I must now use the last two "speed problems" from my files and must ask you to submit some new quickies quick.

Problems

A/S 1. Matthew Chen has a problem that uses less than one-third of a chess board and one-eighth of each army. Place white bishops on a1 and a3 and place black bishops on e1 and e3. By moving each color alternately, such that a black bishop can never take a white one and conversely, and restricting moves to ranks 1 to 4 and files a to e (i.e., to ten black squares), exchange the position of the bishops so that the black bishops end up on a1 and a3 and the white bishops on e1 and e3.

A/S 2. Emmet Duffy asks the following geometry question. An isosceles triangle has a bisector of one of the two equal angles that is 6 inches long. If the base is 5 inches, without using trigonometry, find the length of the two equal sides.

A/S 3. Steve Chilton asks an interesting question about primes. Write the prime numbers and take successive absolute differences

2 3 5 7 11 13 17 19 23 ...
 1 2 2 4 2 4 2 4 ...

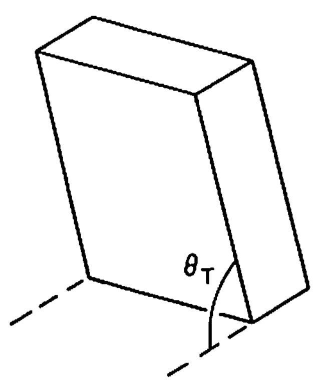
Note that the second row starts with a 1. Next repeat the process of taking successive absolute differences obtaining

2 3 5 7 11 13 17 19 23 29 ...
 1 2 2 4 2 4 2 4 6 ...
 1 0 2 2 2 2 2 2 ...
 1 2 0 0 0 0 0 ...
 1 2 0 0 0 0 ...
 1 2 0 0 0 ...
 ...

The problem is to prove or disprove that the series of leading 1's continues forever.

A/S 4. John Fogarty has sent us the "rug puzzle." You want to put wall-to-wall carpeting into a room that is 9x12 feet. You have two pieces of carpet, one 10x10 and the other 1x8. These do add to the correct square footage, but obviously the 10x10 must be cut. The challenge is to devise *one continuous* cut through the 10x10 piece such that the two resulting pieces will exactly fit the 9x12 area with one gap left over into which the 1x8 remnant can fit to complete the job.

A/S 5. Our last regular problem is reprinted from the M.I.T. Physics Department student newsletter edited by Minn Chung:
On one rainy day Fat Timothy was riding a donkey in the countryside ten miles away from the nearest shelter when he was caught in a strange rain which had a uniform mass density and fell straight down. Worried about being exposed to this strange precipitation, Timothy rode the faithful donkey as fast as he could to the nearest shelter. Assuming that the speed of the donkey was uniform and sufficiently high, find:
1. The amount of rain which fell on Timothy as a function of θ_T, the angle Timothy's body makes with the ground.
2. The minimum and maximum amount of rain which might fall on Timothy.
3. The ideal situation in which Timothy would be wet least.
Approximate Timothy's body as a box:



Speed Department

SD1. Daniel Seidman wants you to find the next term in each of the two (related) sequences.
3 4 8 9 14 23 __
2 9 10 18 20 28 __

SD2. We close with an offering from Smith Turner. Financial types have a rule for how long it takes money at compound interest to double [assuming the interest is compounded yearly—ed.]: years equals 72 divided by the interest rate. For example, at 6 percent, 72/6 = 12 years. What is the mathematical basis for this "rule" and how good an approximation is it?

Solutions

M/A 1. Can the following contract be made against any defense?

North: ♠ 7,3,2 ♥ 5,4,2 ♦ A,10,9,7,6 ♣ 3,2

West: ♠ K,6,4 ♥ K,J,7,6 ♦ K,J ♣ K,Q,J,5

East: ♠ 8 ♥ 9,8 ♦ 8,5,4,2 ♣ 10,9,8,7,6,4

South: ♠ A,Q,J,10,9,5 ♥ A,Q,10,3 ♦ Q,3 ♣ A

The bidding:

South:	*West*	*North*	*East*
1 spade	double	2 diamonds	-
3 spades	-	4 spades	5 clubs
5 spades	double	-	-

We begin with a solution from Amy Lowenstein: The only way West can defeat South is by sacrificing his ♦K, ♠K, and all his club honors; and West must lead the ♦K or else South has all the timing he needs. If West leads anything but the ♦K, South has time to knock out the ♠K and still get to dummy with the ♦A to run diamonds. The probable scenario goes like this: West leads ♦K, South takes with dummy's ♦A, than leads ♦Q and ♣A on which West must take care to throw an honor and not the ♣5. South will try to make dummy's ♠7 an entry by leading the ♠Q, but West will duck. South will lead the ♠J, but West will still duck; so South will then take the ♠A, felling West's ♠K. At this point South should run off two more spades. East should throw diamonds on the spades, and then throw clubs; East must keep both his hearts to avoid West's possibly being end-played by South with the ♥3 to West's ♥7. On the run of the spades, West must throw both club honors. Finally South will play the ♥10, which West will win with the ♥J, but West can get out with the ♣5 to his partner's ♣10. South can ruff or not, but eventually West must make his ♥K. If South has not ruffed the club trick just mentioned, South's three losers are the ♥J, ♥K, and ♣10. Even if South ruffs that club, he still winds up losing the ♥K and ♥7 as the other two losers besides the ♥J, so he is still down one.

Also solved by Gardner Perry, Randy Haskins, John Schindler, John Bobbit, Ronnie Selbst, John Woolston, Richard Hess, Joel Feil, Matthew Fountain, Robert Bart, Winslow Hartford, Warren Himmelberge, and the proposor, Frank Model.

M/A 2. Given 39 balls of which 38 are identical in weight but the 39th is either heavier or lighter than the others, isolate the "odd" ball with only four weighings using a balancing scale. The weighings should also establish if the odd ball is heavier or lighter than the others.

T. Landale sent us a well-organized solution, and Martin Langerveld conjectures that with n weighings the maximum number of balls is $3^{(n-1)} + 3^{(n-2)} + \ldots + 3^1$.

Mr. Landale's solution follows:

The solution is easier to understand if the most common "end game" is described first. This game is with three balls, it being known from previous weighings that (1) one of the three, but not which one, is the odd ball; and (2) for each of the three balls, whether it is heavier or lighter than the others. On the scale compare two balls, each of which is known to be heavier or lighter, if odd. If the scale does not balance, the odd ball is the heavier or lighter one, as the case may be. If the scale does

Weighing	Previous Result (Balance)	(Scale position)	Balls each side	Balls on scale (Left)	(Right)	Solution
A			13	1-13	14-26	
B	if (A) ≠		9	1-6, 14-16	7-12, 17-19	
C	if (B) ≠	& (B) as (A)	3	1, 2, 17	3, 4, 18	
D	if (C) ≠	& (C) as (A)	1	1	2	* 1, 2 or 18
D	if (C) ≠	& (C) not as (A)	1	3	4	* 3, 4 or 17
D	if (C) =		1	5	6	* 5, 6 or 19
C	if (B) ≠	& (B) not as (A)	3	7, 8, 14	9, 10, 15	
D	if (C) ≠	& (C) as (A)	1	7	8	* 7, 8 or 15
D	if (C) ≠	& (C) not as (A)	1	9	10	* 9, 10 or 14
D	if (C) =		1	11	12	* 11, 12 or 16
C	if (B) =		3	20-22	23-25	
D	if (C) ≠	& (C) as (A)	1	23	24	* 23, 24 or 25
D	if (C) ≠	& (C) not as (A)	1	20	21	* 20, 21 or 22
D	if (C) =		2	13, 26	(1, 2)**	13 if (D) as (A) 26 if (D) not as (A)
B	if (A) =		9	27-35	(1-9)**	
C	if (B) ≠		3	27-29	30-32	
D	if (C) ≠	& (C) as (B)	1	27	28	27, 28 or 29
D	if (C) ≠	& (C) not as (B)	1	30	31	30, 31 or 32
D	if (C) =		1	33	34	33, 34, or 35
C	if (B) =		3	36-38	(1-3)**	
D	if (C) ≠		1	36	37	* 36, 37 or 38
D	if (C) =		1	39	(1)**	39 as weighed

balance, the odd ball is the third ball, and it is heavier or lighter as already known. Now proceed to the full solution, where an asterisk (*) signifies this common "end game."

Identify the 39 balls as 1 through 39 and the four weighings as A through D, and proceed according to the table at the top of this page. (At the points marked with the double asterisk (**), use any balls known to be not odd.)

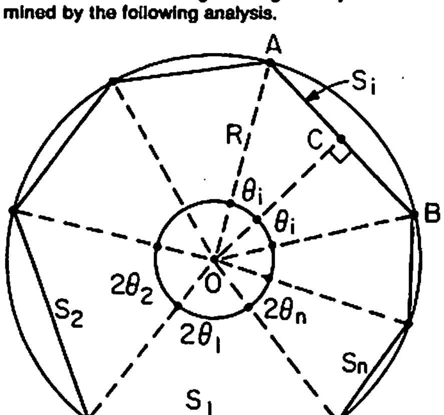
Clearly this is not a unique solution. For example, (D) following (A) ≠, (B) =, and (C) = can be solved simply by weighing (13) or (26) against one ball known not to be odd. Also, (B) following (A) = can lead to solution weighing six balls each side: 27-32 vs. 33-35 plus three balls known not odd; there are other arrangements which will solve (B) following (A) = as well.

Also solved by Robert Bart, Matthew Fountain, Richard Hess, John Woolston, Thomas Peterson, Joe Fell, Ronnie Selbst, Ken Arbit, John Bobbit, John Rule, Frank Carbin, Emmet Duffy, Norman Spencer, Walter Smith, Edgar Rose, and the proposer, Arun Trikha.

M/A 3. Given the lengths of the n sides of an irregular polygon, how should the sides be arranged and what should the angles be in order to maximize the area?

Harry Zaremba sent us the following carefully drawn solution:

The maximum area that can be enclosed by an irregular polygon occurs when each vertex lies on a circle circumscribing the polygon. The radius of the circle and polygonal area remain unchanged for any peripheral combination of the given sides of the inscribed polygon. The maximum area which can be confined within a given n-gon may be determined by the following analysis.



In the figure shown,

R = radius of the circle circumscribing the n-gon.

S_i = length of the ith side of the n-gon. ($i = 1,2,3, \ldots, n$).

$2\theta_i$ = angle subtended by S_i at center O of the circle.

For the indicated ith triangle AOB, OC = $R \cos \theta_i$ and

$S_i = 2R \sin \theta_i$, or

$R = S_i/(2 \sin \theta_i)$ (1)

The area of AOB is

$A_i = (S_i/2)R \cos \theta_i = (S_i^2/2) \cot \theta_i$.

Therefore, the total area of the polygon becomes

$A = \frac{1}{4}(S_1^2 \cot \theta_1 + S_2^2 \cot \theta_2 + \ldots + S_n^2 \cot \theta_n)$. (2)

Also from the figure, the sum of one-half of the central angles is

$\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n = \pi$. (3)

Using equation (1), the ratio of the expression for the ith side to that for side S_1 yields

$S_i/S_1 = (\sin \theta_i)/(\sin \theta_1)$, or

$\theta_i = \sin^{-1}(S_i/S_1) \sin \theta_1$ (4)

Substitution of θ_i into (3) for $i = 2,3, \ldots, n$ results in (5)

$$\theta_1 + \sin^{-1}\left(\frac{S_2}{S_1}\sin \theta_1\right) + \sin^{-1}\left(\frac{S_3}{S_1}\sin\theta_1\right) + \ldots + \sin^{-1}\left(\frac{S_n}{S_1}\sin \theta_1\right) - \pi = 0$$

The solution to equation (5) can be obtained readily by iteration methods to any practical accuracy with the use of a computer or programmable calculator. With θ_1 determined, all other angles θ_i, maximum area A, and radius R of the circle can be evaluated from equations (4), (2), and (1), respectively. For example, when n = 5 and sides S_1 to S_5 equal 5, 4, 3, 2, and 1, respectively, the central half angles are:

θ_1 = 66°55′2.6″
θ_2 = 47°23′15.9″
θ_3 = 33°30′6.7″
θ_4 = 21°35′27.1″
θ_5 = 10°36′7.7″

and the area A = 13.60499 square units and the radius R = 2.717567.

Also solved by Matthew Fountain, Robert Bart, and the proposer, Irving Hopkins.

M/A 4. In how many ways can seven people be seated at a round table so that no person sits next to the same pair (unordered) of people twice?

Several readers pointed out that for n people, at most

(n - 1)(n - 2)/2

arrangements are possible. Thus for n = 7 we obtain an upper bound of 15. Apparently the exact value is not known. Most responders found 10 arrangements, but Ken Arbit found the following 11.

1, 7, 6, 5, 4, 3, 2
1, 5, 6, 4, 3, 7, 2
1, 5, 4, 6, 3, 2, 7
1, 4, 7, 5, 3, 6, 2
1, 7, 4, 5, 3, 2, 6
1, 6, 7, 4, 3, 5, 2
1, 6, 4, 7, 3, 2, 5
1, 3, 6, 7, 5, 4, 2
1, 3, 4, 2, 6, 5, 7
1, 6, 3, 7, 5, 2, 4
1, 3, 7, 6, 2, 5, 4

Also solved by Matthew Fountain, Richard Hess, Gardner Perry, and Harry Zaremba.

M/A 5 We begin with two integers m and n such that $2 \leq m \leq n \leq 99$.
We tell Mr. P the product mn, and we tell Ms. S the sum m + n. the following conversation then takes place:
Mr. P: "I don't know the two numbers."
Ms. S: "I knew you didn't know. I don't know either."
Mr. P: "Now I know the numbers."
Ms. S: "Now I know, too."
What are m and n?

Mark Lively had little trouble with this one but admits surprise at the high level of civilization now found at M.I.T.

'*Mr. P:* "I don't know the two numbers" ' precludes m and n from being both prime, since their product would then be factorable in one way only, as the two primes, and the answer be transparent to Mr. P.

'*Ms. S:* "I knew you didn't know, I don't know either" ' precludes m and n from summing to any number equal to the sum of two primes. For example, "4,4" would be revealed to Ms. S as "8". The revelation of an "8" to Ms. S does not preclude "3,5" which would be communicated to Mr. P as "15", revealing the answer to Mr. P. Similarly, all m and n's summing to greater than "54" are precluded, since they would allow "53,n" and a transparent answer for Mr. P. Ms. S's number must be "11,17,23,27,29,35,37,41,51, or 53."

'*Mr. P:* "Now I know the numbers" ' precludes the product from having more than one pair of factors $f_i g_i$ whose sum appears in the above list. Thus Mr. P's number cannot be "30" since that allows "2,15" and "5,6", both of whose sums "11" and "17" appear in the list.

'*Ms. S:* "Now I know, too" ' precludes the sum from having two items not eliminated in the above. The sum "11" permits "2,9" (product "18"); "3,8" (product "24"); "4,7" (product "28"); and "5,6" (product "30"). Since only the latter combination was eliminated by Mr. P's statement, the first three are eliminated by Ms. S's statement.

The answer is 4,13. Your statement of the problem seems longer (and more civilized) than I would expect from my fellow students. A shortened version follows:
Ms. S: "You don't know the answer."
Mr. P: "Now I do."
Ms. S: "So do I."

Also solved by Sidney Shapiro, Carl Goodwin, Elliot Roberts, John Bobbit, Richard Hess, Matthew Fountain, Ken Arbit, Ronnie Selbst, John Woolston, Amy Lowenstein, Douglas Fink, Michael Jung, Frank Norton, Win Haviland, Bryan Sayrs, Edwin McMillan, and Robert Bart.

Better Late Than Never

FEB 1, Feb 3, and FEB 5. Allen Tracht has responded.

Proposers' Solutions to Speed Problems

SD1. The last number in the first sequence is 25; the numbers are the partial sums of the digits of π. The required number in the second sequence is 29, the numbers being the partial sums of the digits of e.

SD2. If i is the interest rate and P is the product of the interest rate and the number of years required (the "rule" says P ≈ 72), then

$$(1 + i/100)^{P/i} = 2$$

giving

$$P = \frac{i \log(2)}{\log\left(1 + (i/100)\right)}$$

As i approaches 0, P approaches 100 ln(2) ≈ 69.3. Moreover, P increases very slowly with i, equaling 72 at 7.85 percent and only 74.7 at 16 percent. The 72 is selected because it occurs in mid-range of usual rates and is divisible by so many integers. The rule is accurate to within 5 percent for interest rates from 2 to 18 percent.