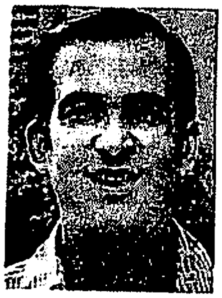
Puzzle Corner
Allan J. Gottlieb

Why Is It Always 8:18?



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Winthrop Leeds, '39, a frequent contributor to Puzzle Corner, sent me a charming monograph of his work on Pythagorean triples. While I was intrigued by the appearance of several problems from Puzzle Corner, I was considerably more fascinated by a philosophical point made by Dr. Leeds. He recalled how, after he had made a long and painstaking classification of all triples
$A^2 + B^2 = C^2$ with $0 < A,B,C < 1000$
he received a letter from a friend, Dr. D. C. Lewis, giving a simple formula from which all these triples can be obtained easily. After some initial dismay, Dr. Leeds realized that it was searching for the triples that made finding them so rewarding. "Things handed to you . . . never seem so precious as the treasures earned by the sweat of the brow."

I am always amazed to find how far Puzzle Corner travels. The well-known Japanese puzzle creator Nob. Yoshigahara has sent me several of his mazes and other challenges, and some of these will appear in future issues.

Problems

JUL 1 We begin with an arithmetic progression of bridge problems from Douglas Van Patter:
You wish to maximize your chances in playing this nine-card suit:
Dummy: A K 10 x
Declarer: x x x x x
You play the A from the Dummy, and an honor falls on your right. It is well known in bridge circles that the odds are now nearly 2:1 in favor of finessing West for the other honor (Rule of Restricted Choice), provided that East is known to play either the Q or J at random from a doubleton Q-J holding. The problem is to consider what happens as the combined holding shrinks from nine cards to five while still maintaining the two honors (the limiting case is A K 10 in Dummy) and calculate the exact odds for the finesse in each case (nine cards, eight, etc.).

JUL 2 Roy Sinclair needs to land on the moon softly, so he writes:
A rocket containing 60 gallons of fuel approaches the moon (gravity is 5 ft./sec./sec.) tail-first with an initial velocity of 50 ft./sec. when at an altitude of 500 ft. Each second thereafter an integral number of gallons of fuel may be burnt, causing an upward acceleration of x ft./sec./sec., where $x = 2f - 5$. Find the sequence of one-second fuel burns that maximizes the maximum f while yielding a soft landing.

JUL 3 Winslow Hartford asks a "timely" problem:
Watchmakers all over the world display a symmetrical watch face in which the hands are set at 8:18. Although this was often believed to represent the hour of Lincoln's death, the custom actually is much older. Another idea is that the arrangement gives the jeweler a maximum amount of advertising space, but this doesn't hold water either. I would like to submit the idea that this watch-face division goes back to the "Golden Mean" of the ancient Greeks, in which the most esthetically pleasing division of space was believed to be that in which the larger part occupied $[(\sqrt{5}-1)/2]$, or approximately 72 percent of the total area. Test this idea. Is there an arrangement of the hands near 8:18 that is both symmetrical and in accord with the Golden Mean? If not, what is the time at the nearest symmetrical point and at the Golden Mean point?

JUL 4 Smith Turner wants you to find a rational number (other than 41/12) such that its square, when increased or decreased by 5, remains a square.

JUL 5 Our last regular problem first appeared in *Technology Review* in 1941 as part of an advertisement for Calibron Products:
Suppose that some new type of photographers' light bulbs undergoing a life-test for one week burned out as follows:
Sunday: ½ of the bulbs + ½ of a bulb,
Monday: ⅓ of the bulbs left + ⅓ of a bulb,
Tuesday: ¼ of the bulbs left + ¼ of a bulb, and so on progressively until
Saturday: ⅛ of the bulbs left + ⅛ of a bulb.
Assuming that there is only one filament in each bulb, what is the least number of bulbs that could have been left when the test ended? If the fractions had progressed in reverse order (starting with ⅛ of the bulbs + ⅛ of a bulb on Sunday), would the final result have been the same? Why?

Speed Department

JUL SD 1 My old classmate Chet Sandberg has the following "speed puzzle for college students":

```
   S E N D
   M O R E
 ---------
 M O N E Y
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What digits do the letters represent?

JUL SD 2 Irving Hopkins has a problem that's as speedy as your calculator: using radians, find approximate solutions to:
$u = \sin(v)$
$v = \cos(u)$.

Solutions

We begin with the solution to **JAN 3**, and an apology from the editors for the error which forced its omission from its proper place in the previous issue of *Technology Review*.

JAN 3 What is the minimal 3 x 3 magic square composed solely of prime numbers? (For this problem, 1 is considered prime and a minimal square is one whose (equal) row, column, and diagonal sums are minimal.)

Matthew Fountain used a new toy and an old book to help him solve this one; he writes:
I now have my own IBM personal computer, and the first program I ran on it was your JAN 3. It was a real learning experience. The instructions that come with the computer are fairly complete, but they are not organized so a rank beginner finds them easy to follow. However, the computer does a good job of pointing out errors.
The following magic square of primes is attributed to H. E. Dudeney by W. W. Ball in his book, *Mathematical Recreations and Essays:*

7	73	31
61	37	13
43	1	67

It is the minimal magic square of primes when 1 is defined as a prime. The general form of magic squares of the third order is:

A	3C − A − B	B
C − A + B	C	C + A − B
2C − B	A + B − C	2C − A

To avoid squares that are rotations and reflections of others, one may impose the restriction $A<B<C$. Also $C<A+B$, to assure that $A+B-C$ is positive. If A, B, and C are restricted to primes less than 250, there are 27 independent magic squares of primes of order three. Dudeney's is the only one of these containing 1. The minimal magic square of primes where 1 is not a prime is:

17	113	47
89	59	29
71	5	101

Also solved by Charles Rivers, Edwin McMillan (who found a 1913 (!) reference), George Ropes, Avi Ornstein, Richard Hess, Emmet Duffy, Harry Zaremba, Winslow Hartford, John Woolston, William Katz, Lyndon Welch, and the proposer, Gertrude Fox.

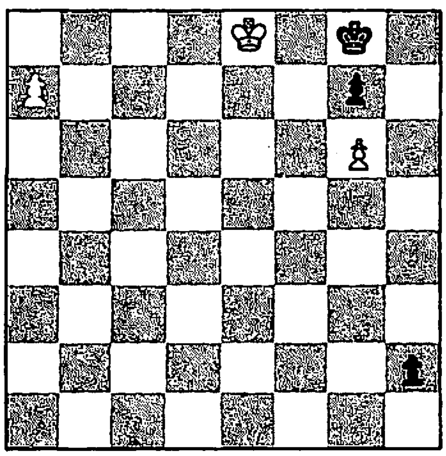
Now we return to schedule with solutions to the problems published in the February/March issue:

FEB 1 On an island there are only two trees, A and B, and the remains of a gallows. According to an old map, treasure may be found by following these directions: start at the gallows, pace to A, turn 90° to the left, pace an equal distance, and drive a stake. Return to the gallows, pace to B, turn 90° to the right, pace an equal distance, and drive another stake. Treasure is buried at a point half-way between the two stakes. A treasure hunter, coming to the island, found the two trees, but all vestiges of the gallows were gone. None the less, he found the treasure. How did he do it?

The following solution is from Edgar Rose:
The treasure hunter paced from tree A to the midpoint between trees A and B, made a right turn (90°), paced a distance equal to that between A and the midpoint, stopped, and dug up the treasure.
Proof: place rectilinear coordinates on the map, with tree A at (0,0), and tree B at (B,0). The gallows would have been at any point (a,b). Then following the original instructions, stake A would be at (b, -a) due to the transposition of (a,b) into the clockwise adjacent quadrant. Similarly, but using a transposition into the counterclockwise adjacent quadrant and translation of coordinate origin from (0,0) to (B,0), stake B would be at (B-b, a-B). Since the midpoint between any two points (x_1, y_1) and (x_2, y_2) is $[\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)]$, it follows that the treasure is located at $(\frac{1}{2}B, -\frac{1}{2}B)$ and its location is independent of a and b.

Also solved by Buzz Karpay, Michael Jung, Steve Feldman, Raymond Gaillard, John Prussing (who points out that he submitted this same problem to Puzzle Corner and we published it in 1975), Richard Hess, Norman Wickstrand, Harry Zaremba, Christian Marchand, Jeff Oehler, Emmet Duffy, Doug Van Patter, Avi Ornstein, John Venti, William Schoenfeld, Winslow Hartford, Winthrop Leeds, and Harry Garber.

FEB 2 White to move and win:



Many people fell into the carefully set trap, but not John Bobbit, who writes:
White has a winning position but he must play carefully.

1. P—R8 (Q)	P—R8 (Q)
2. Q—N8!	

Both players must queen. White now threatens K—Q7 mate. Note that 2. Q x Q is stalemate. Black can keep this possibility alive by

2.	Q—R7
3. Q—Q8	

White again threatens K—Q7 mate.

3.	Q—R5
4. Q—Q5 ! ck	K—R1
5. Q—R8 !	

White again threatens K—Q7 mate. But note that Black can no longer force Q x Q stalemate, because the Black king is now in the rook file. Instead

5.	Q—R8
6. Q x Q ck	K—N1
7. Q—R7 mate	

After 2. Q—N8!, Black cannot reply with check. If

2.	Q—K5 ck
3. K—Q7	

leads to mate, Black cannot work to interpose his queen. If

2.	Q—B6
3.K—K7 ck	Q—B1
4. Q x Q mate	

Any other move by Black, of course, could be met by 3. K—Q7 mate or K—K7 mate. Note that 2. Q—N8! is the only move that works. If

2. Q—Q8	Q—N2

leads to a draw. If

2. Q—B8	Q—R6
3. Q—Q8	Q—K3 ck
4. Q—K7	

Black is forced to move

4.	Q x Q

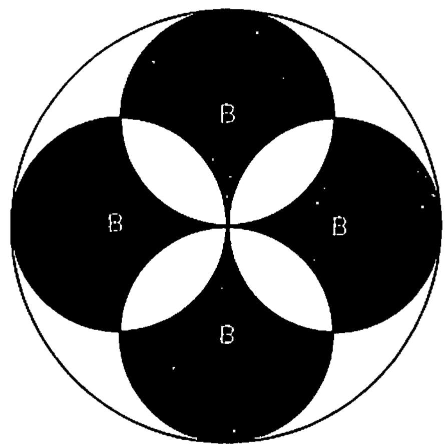
If

2. Q—R2 ck	K—R1

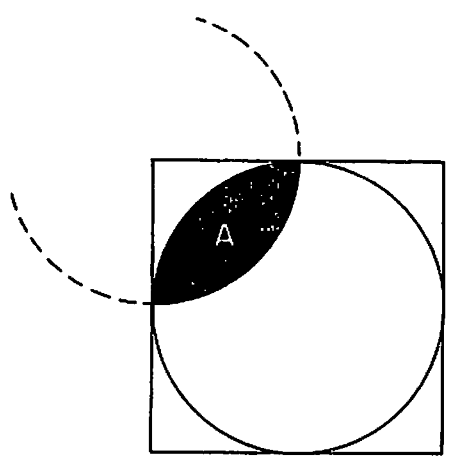
Black will be able to continually check, or trade queens, or position his queen on the first rank. Any of these leads to a draw.

Responses were received from Raymond Gaillard, Richard Hess, Winthrop Leeds, Smith Turner, Steven Kiliment, Winslow Hartford, George Farnell, Edgar Rose, G. Sharman, T. Burtness, Robert Close, Michael Bercher, Norman Wickstrand, and the proposer, Stuart Schulman.

FEB 3 A trained chimpanzee can hit the patterned dartboard with one out of every two throws, on the average. What is the probability that on a given throw the dart will hit the shaded region?



The following carefully drawn solution is from Buzz Karpay: Let the radius of the large circle be 2, making the radius of a small circle 1.

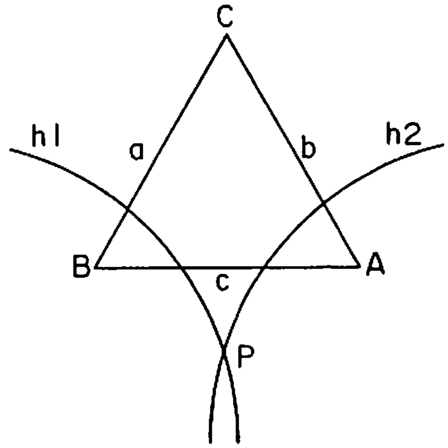


The area of the intersection of two circles (A) can easily be found:
$A = \pi/4 - (1 - \pi/4) = \pi/2 - 1.$
Thus the area of each of the four dark portions of the dartboard can be expressed as:
$B = \pi - 2(\pi/2 - 1) = 2.$
The total shaded area is 4B = 8, and the area of the large circle is 4π. The percentage of the large circle which is shaded is $8/4\pi = 2/\pi$. Since only every other throw of the dart hits the board, the probability that on a given throw the dart will hit the shaded region is $(2/\pi)/2 = 1/\pi = 0.3183$. . .

Iri Smith notes that we did not specify the probability distribution of the chimp's throws and presents solutions for several distributions different from the uniform distribution used by Karpay and most other readers. Also solved by Norman Wickstrand, Winslow Harford, Richard Hess, John Bobbit, Harry Garber, Jeff Oehler, Ken Arbit, Dave Keller, Harry Zaremba, Raymond Gaillard, Marshall Fritz, David Lukeres, Mary Lindenberg, Richard Farber, Paul Mogolesko, Greg Huber, Frank Carbin, Philip Burstein, Michael Jung, and the proposer, John Prussing.

FEB 4 Locate the point P in the plane of a given triangle ABC such that triangles PAB, PBC, and PCA have equal perimeters.

Only Richard Hess found a geometric construction:



Let the sides of triangle ABC satisfy $a \leq b \leq c$. Form the hyperbola h1 intersecting side a and having the property that the distance from h1 to C exceeds the distance from h1 to B by $c - b$. Next form the hyperbola h2 intersecting side b and having the property that the distance from h2 to C exceeds the distance from h2 to A by $c - a$. The intersection of these two hyperbolas gives the desired point P.

Also solved by Harry Zaremba, Winthrop Leeds, and Winslow Hartford.

FEB 5 What can be said about the time periods of surface orbits of spheres of the same density but of different sizes?

Everyone agrees that, for a given density, a surface orbit has a period independent of the radius. The following is from Michael Jung:

Start with Newton's Law:

$F = ma = m(4R\pi^2)T^2$,

where R is the radius and T the period. Then recall the law of gravitation,

$F = GmdV/R^2$,

where G is constant, d is density, V is volume. Thus

$GdV = 4\pi^2R^3/T^2$.

Since $V = (4/3)\pi R^3$, we obtain

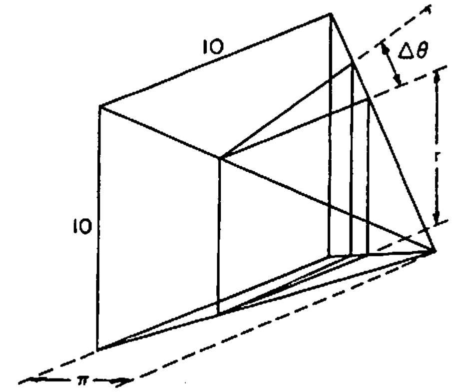
$Gd/3 = \pi/T^2$. Or $T = (3\pi/Gd)^5$

which is clearly independent of R.

Also solved by Emmet Duffy, Harry Zaremba, John Prussing, Richard Hess, and David Lukeres.

Better Late Than Never

OCT 2 Emmet Duffy notes that the second figure given does not quite agree with the one he submitted. As printed the elementary section is shown as a wedge, whereas it should be two parallel r x r squares delta theta apart.



OCT 4 Emmet Duffy, Alan Prince, and a third correspondent whose last name I cannot read note that the published solution is wrong, as it incorrectly generalizes two-dimensional geometry from three-dimensional geometry. Mr. Duffy submitted the following solution. In doing so he notes that this is a "most difficult problem to solve and even more difficult to explain"; he corrected himself twice before sending me the following:

Construct a tetrahedron with four equilateral triangles as faces, with R as the length of each side of a triangle. Then four spheres with radius R can be placed with the center of a sphere at a tetrahedron vertex and the surface of the sphere will pass through the other three vertices of the tetrahedron. The volume of the tetrahedron will be common to all four spheres and will have a base, $\sqrt{3}\,R^2/4$. The height will be $R\sqrt{2}\sqrt{3}$. Volume is then base times height times $1/3 = R^3\sqrt{2}/12 = 0.1178511302R^3$. Additional volume must be added to the tetrahedron to find volume common to all four spheres. Some of it is found by extending outward the planes of any three faces of the tetrahedron so that they intersect the surface of a sphere, forming a spherical triangle. The dihedral angle of a tetrahedron has a cosine equal to $1/3$, making the angles 70.52877937°. Each of the three angles of the spherical triangle will be this value, making the sum 211.58633811°. The spherical excess, E, is then the sum of the angles minus 180°, or 31.58633811°. The area of the spherical triangle is given by $\pi R^2E/180$. Multiplying by R/3 will give the volume of the solid angle, that is, the volume of the tetrahedron plus the volume added to one side, totalling $\pi R^3E/540$, or $0.1837618661R^3$. Subtracting the volume of the tetrahedron leaves $0.0659107359R^3$ which is the volume added to one side of the tetrahedron. Multiplying this figure by four and adding the volume of the tetrahedron results in $0.3814940738R^3$. This is the answer I submitted and is also the published answer, but it is incomplete.

Referring to Fig. 2 (opposite), the volumes added to two sides of the tetrahedron are shown in outline as EHF and FJG. Note that two wedges HFI and IFJ of the same shape must be added at all six edges of the tetrahedron. Wedge HFI of Fig. 2 has a base which as shown in Fig. 3 as LMN, which is a segment of a circle with radius R. The tilted side of the wedge, as shown in Fig. 4, is segment PQS of a circle with radius $\sqrt{3}\,R/2$. The angle between segments is 35.2644 degrees.

The evaluation of the volume of a wedge will be difficult unless some assumptions are made which will have negligible effect. The base of the wedge is shown in Fig. 5 and the tilted portion in Fig. 6. If a vertical plane in Fig. 5 is passed through the wedge at a distance x from the Y axis, it will intercept AB which is $(R^2 - x^2)^{1/2} - \sqrt{3}\,R/2$. Negligible error (less than +2 percent) will be made if it is assumed that the plane makes an intercept of a right triangle with

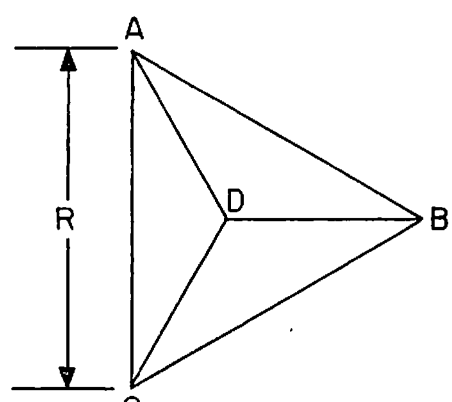


Figure 1: Top view of tetrahedron.

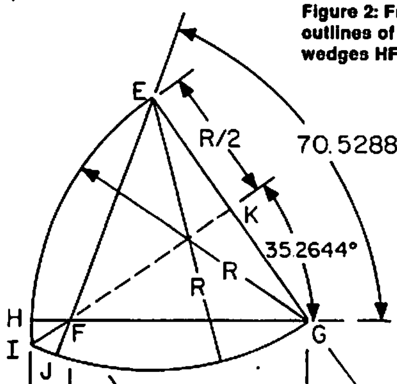


Figure 2: Front view of tetrahedron showing outlines of solid angles EFGJ and GEHF and wedges HFI and IFJ.

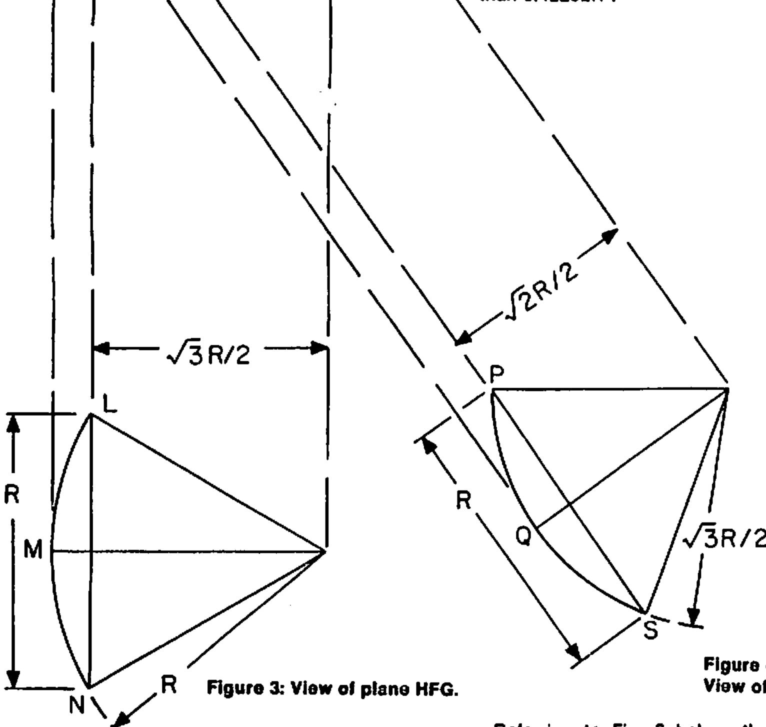


Figure 3: View of plane HFG.

Figure 4: View of plane IFK.

base AB and hypotenuse at an angle 35.2644°. This angle will be called a. This approximation will be slightly more than the correct value.

The triangle with base AB will have height AB tan a and area (AB^2 tan a)/2. Integrating from x = 0 to x = 0.5R and multiplying by 2, the volume is given by:

$$V = (2 \tan a)/2 \int_0^{0.5R} (AB)^2 dx.$$

As AB = $(R^2 - x^2)^{1/2} - \sqrt{3}\,R/2$, and tan a = $\sqrt{2}/2$, then:

$$V = \sqrt{2}/2 \int_0^{0.5R} [R^2 - x^2 - \sqrt{3}R(R^2 - x^2)^{1/2} + 3R^2/4]\,dx.$$

From a table of integrals,

$$\int (R^2 - x^2)^{1/2} = x/2(R^2 - x^2)^{1/2} + (R^2/2) \text{ arc sin } (x/R) + C.$$

Then

$$V = \sqrt{2}/2\Big(R^2x - x^3/3 - \sqrt{3}R\big[(x/2)(R^2 - x^2)^{1/2} + (R^2/2) \text{ arc sin } (x/R)\big] + 3R^2x/4\Big)\Big|_{x=0}^{x=0.5}$$

This yields V = $0.0034531R^3$, and 12V = $0.0414372R^3$, giving a total volume slightly less than $0.42292R^3$.

Referring to Fig. 6 below, the volume of the wedge can be approximated, with a value slightly less than correct. A vertical plane is passed through the solid wedge, intercepting the tilted side AB which equals $(3R^2/4 - x^2)^{1/2} - \sqrt{2}R/2$. A slightly-less-than-correct answer will be obtained by assuming that AB is the hypotenuse of a right triangle. The vertical plane will then intercept the wedge forming a right triangle with base ABcos a and

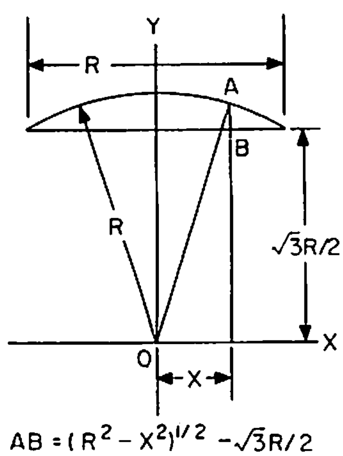


Figure 5.

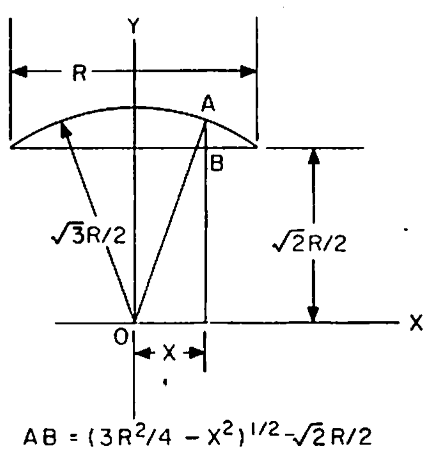


Figure 6.

height ABsin a; its area will be $[(AB)^2 \sin a \cos a]/2$. Integrating from x = 0. to x = 0.5R and multiplying by 2, the volume is

$$V = \sin a \cos a \int_0^{0.5} (AB)^2 dx.$$

Given that $AB = (3R^2/4 - x^2)^{1/2} - \sqrt{2}R/2$ and $\sin a = \sqrt{3}/3$, $\cos a = \sqrt{2}/\sqrt{3}$, then:

$$V = \sqrt{2}/3 \int_0^{0.5R} [3R^2/4 - x^2 - \sqrt{2}R(3R^2/4 - x^2)^{1/2} + R/2]\, dx$$

$$V = \sqrt{2}/3 \left\{ 3R^2x/4 - x^3/3 - \sqrt{2}R\,[(x/2)(3R^2/4 - x^2)^{1/2} + 3R^2/8(\arcsin \frac{x}{\sqrt{3}R/2} + R^2x/2 \right\}_0^{0.5R}$$

This yields $V = 0.0032650R^3$, and $12V = 0.03918R^3$. Adding to the value already computed, $0.38149R^3$, gives a total of slightly more than $0.42067R^3$. The required volume is thus between $0.42067R^3$ and $0.42292R^3$.

N/D 1 Robert Bart, Eric Backus, and Douglas Fink found shorter solutions. Mr. Bart's are:

	Flip disc	*Othello-instant*
1	D6	C4
2	D3	E6
3	B5	C6
4	D4	C5
5	F5	

N/D 2 Chris Johnson submitted the following improvement:

In his solution, William Schumacher leaves a hole. It is true that CK and FJ intersect as point D on AB and BJ and GH intersect at point E on AC. But does GEH necessarily intersect AB at point C? For the proof to be complete, this must be shown—viz., that AFG forms an equilateral triangle and that GH therefore bisects and is perpendicular to AF. (Let's call the point of intersection L.) Triangle ADF is isosceles; therefore the line which bisects and is perpendicular to AF is DL which is a segment of GH. QED.

Responses were also received from R. Bart, Eugene Sard, and Arthur Poe.

N/D SD 1 John Kellam writes:

The McGuinness list was incorrect; there are not 110 triangles. Although CIP was missing, two duplicates appeared (CKD with CDK and DIG with DGI), reducing the 110 to 109. There were also four typographic errors (DFP for BFP, BFI for BGI, EFI for EGI, and COP for GOP) which did not affect the count. I'm still wondering why Adam Becker, who proposed such a good speed problem, offered a solution with only 79 triangles.

Responses were also received from L. Steffens and Henry Ferguson.

Proposers' Solutions to Speed Problem

SD2 A programmable calculator can be used, but the proposer, Irving Hopkins, writes that he found his TI-SR50A to be more fun:

Punch	*Display*
1	1
sin	.8414709848
cos	.6663667454
sin	.6181340710
cos	.8149612096
sin	.7276989929
et cetera	
finally, u = sin	.6948196907
v = cos	.7681691567

Besides the (sin, cos) group, Mr. Hopkins finds (sin,tan), (cos,tan), (sin,cos,tan), and (cos,sin,tan) work equally well. But the group (cos,tan) is a surprise, in that the solution comes out

u = .9999060062
v = 1.557085794
u = .0137101028
v = .0137109619

which repeats.