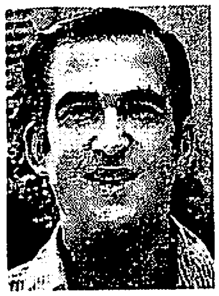
Puzzle Corner
Allan J. Gottlieb

Can You Beat the Chimpanzee?



Allan J. Gottlieb, '67, is associate research professor of mathematical sciences at the Courant Institute of Mathematical Sciences of New York University; he studied mathematics at M.I.T. and Brandeis. Send problems, solutions, and comments to him at the Courant Institute, New York University, 251 Mercer St., New York, N.Y. 10012.

Let me begin by reviewing the sizes of the various problem backlogs. My cup of regular problems runneth over (more than a two-year supply); I have about a half-year supply of "quickies" and bridge problems; and a year's worth of chess puzzles.

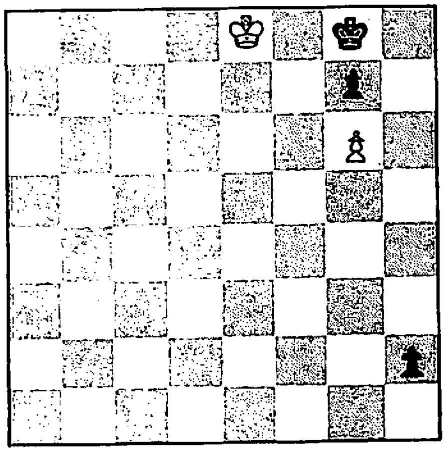
Emmet Duffy comments that about three years ago I indicated that an anthology of problems was being considered. Mr. Duffy then remarks, "What happened? Get going and pay off the mortgage!" What happened was that a reader had at first expressed interest in compiling a collection from Puzzle Corner and then changed his mind. As I have said before, the halls of New York University are not filled with publishers waiting their turns to bid on such an anthology.

Mr. Duffy also expressed an interest in Walter Penny's solution to the perfect squares problem **1981 JAN 2**. If Mr. Penny will supply his method of solution, I will be glad to forward it to all interested parties.

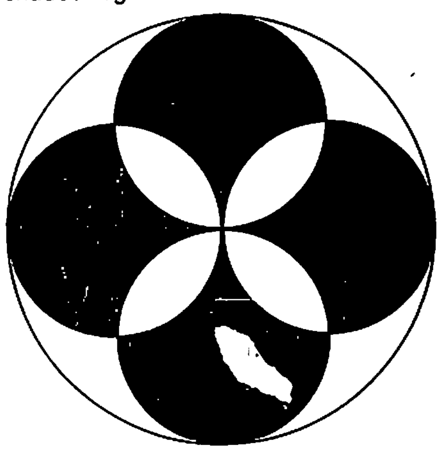
Problems

FEB 1 Winslow Hartford enjoyed the following problem that appeared in *Scientific American* during Martin Gardner's tenure: On an island there are only two trees, A and B, and the remains of a gallows. According to an old map, treasure may be found by following these directions: start at the gallows, pace to A, turn 90° to the left, pace an equal distance, and drive a stake. Return to the gallows, pace to B, turn 90° to the right, pace an equal distance, and drive another stake. Treasure is buried at a point half-way between the two stakes. A treasure hunter, coming to the island, found the two trees, but all vestages of the gallows were gone. None the less, he found the treasure. How did he do it?

FEB 2 Stuart Schulman offers a chess problem in the classic format: White to move and win.



FEB 3 John Prussing's chimpanzee is trained to throw darts at the patterned dartboard below. He can hit the dartboard with one out of every two throws, on the average. What is the probability that on a given throw the dart will hit the shaded region?



FEB 4 John Rule wants you to locate the point P in the plane of a given triangle ABC such that triangles PAB, PBC, and PCA have equal perimeters.

FEB 5 Jerome Taylor thinks our orbital space flights would be more interesting if done just above the surface of the earth. He writes, "What can be said about the time periods of surface orbits of spheres of the same density but of different sizes?"

Speed Department

FEB SD1 Emmet Duffy has only a compass and pencil and needs to draw a circle and mark off a 90-degree arc. How can he do it?

FEB SD2 Here's one from Art Delagrange that should help me decide on presents for my wife Alice (it is late November as this is written):

In the traditional Christmas carol, the young lady receives on the first day a partridge in a pear tree; on the second day two turtle doves and a partridge in a pear tree; and so forth until on the twelfth day she receives 12 drummers drumming, 11 pipers piping, 10 lords a-leaping, 9 ladies waiting, 8 maids a-milking, 7 swans a-swimming, 6 geese a-laying, 5 golden rings, 4 calling birds, 3 French hens, 2 turtle doves, and a partridge in a pear tree. Of what items did she receive most, and of what least?

Solutions

OCT 1 With the hands shown, East opens with a weak two-diamond bid (usually six cards, sometimes only five). South's final contract is six hearts. The opening lead: ♦2. What line of play gives South the best chance of making this slam, given the knowledge that West has three trumps?

North:
♠J 10 5 2
♥8 6 2
♦7 6 5 4
♣K J

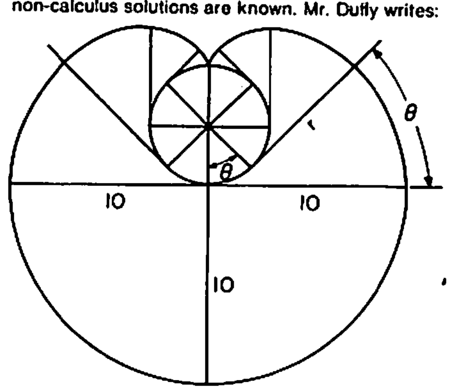
South:
♠A K
♥A K Q 5 4 3
♦A 8 3
♣A 9

Since East has most of the outstanding diamonds, West has more than a 25-percent chance of having ♠Q and ♣Q. In this case, Rev. Joseph Hahn's solution works: Play ♦A, draw trumps in three rounds, lead ♠A and ♠K, lead ♣9 and finesse the ♣J. Next lead the ♠J, discarding the ♣A. West will have to lead either a spade to the ♠10 or a club to the ♣K; in either case the two little diamonds are parked on these two black cards.

Also solved by Barbara Magid, John Rutherford, John Stiehler, Allen Zaklad, Winslow Hartford, Linda Hicks, Mark Oshin, George Holderness, Matthew Fountain, Richard Hess, and the proposer, Doug Van Patter.

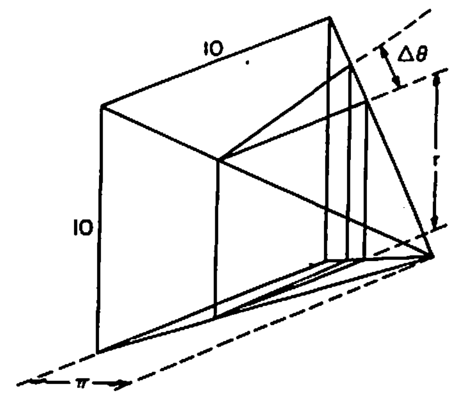
OCT 2 A cow is tethered to a circular silo in such a manner that the distance from the cow's mouth to the fixed tie-point is exactly 10 meters. If the circumference of the silo is exactly 20 meters, how much grazing area does the cow have? (A calculus solution was requested.)

The proposor solved this by treating the silo as an n-gon and taking the limit as n approaches infinity (not quite calculus but in the same spirit). Emmet Duffy's solution reprinted below converts the area of an involute to the volume of a pyramid for which non-calculus solutions are known. Mr. Duffy writes:



The grazing area shown in the figure is a semicircle with radius 10 meters and two involutes. As the rope winds around the silo, the radius varies linearly from 10 to 0 as the angle varies from 0 to π. At a radius r, the area of an elementary sector is $r^2\Delta\theta/2$. The area of the involute is the summation of $r^2\Delta\theta/2$, but the summation of $r^2\Delta\theta$ for the involute is a pyramid with a square base of side 10 in meters and a height of π. Volume is $100\pi/3$. Then the summation of $r^2\Delta\theta/2$ is $100\pi/6$, which is the area of the involute. The total area is then

$2 \times 100\pi/6 + 100\pi/2 = 250\pi/3 = 261.9$ square meters.



Also solved by Richard Hess, Winthrop Leeds, Matthew Fountain, Norman Wickstrand, Harry Zaremba, and Mary Fenockettl.

OCT 3 Find an integer having the property that moving the rightmost digit to the leftmost slot yields a number exactly nine times the original number.

John Woolston and Edwin McMillan generalized the problem, and the following solution is from Harry Zaremba:

Let N be the required integer which in separate digit form is

$N = 10^n a_n + 10^{n-1} a_{n-1} + 10^{n-2} a_{n-2} + \ldots + 10^2 a_2 + 10 a_1 + a_0$.

By moving a_0 to the extreme left position, the new number M will be

$M = 10^n a_0 + 10^{n-1} a_n + 10^{n-2} a_{n-1} + \ldots + 10^2 a_3 + 10 a_2 + a_1$.

From the condition of the problem,

$M = 9N.$ (1)

If we substitute the digital form of the numbers into the latter relation and rearrange the terms so that only the a_0 terms remain on the right, we get

$10^{n-1} a_n + 10^{n-2} a_{n-1} + \ldots + 10 a_2 + a_1 = (10^n - 9)/89 \cdot a_0.$ (2)

The fractional factor on the right of the equation will be an integer when n = 43, and the relation (1) will be satisfied when $a_0 = 9$. Therefore, the integer expressed by equation (2) is a 43-digit number equal to 1,011,235,...,022,471, whe[r]e $a_{43} = a_1 = 1$. Attaching $a_0 = 9$ to the [left] of the above number, the required value of N is a 44-digit number:

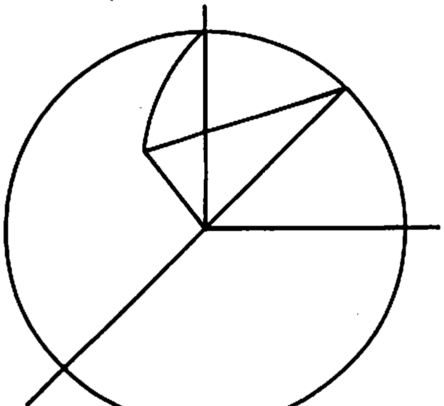
10,112,359,550,561,797,752,808,988,764,044,-943,820,224,719.

Also solved by Matthew Fountain, Avi Ornstein, Harry Zaremba, Emmet Duffy, John Bobbitt, Bruce Garetz, Susan Henrichs, Norman Spencer, Mark Oshin, John Rule, Donald Savage, Eric Wilisky, Winslow Hartford, R. Mulfikin, Winthrop Leeds, V. Caravito, Richard Hess, Mary Fenockettl, and the proposor, Bill Bezdecheck.

OCT 4 In the Ballantine Beer logo, in which three circles of equal radius are placed so that the center of each circle lies on the boundary of the other two, the area of the region of intersection is $r^2(\pi - \sqrt{3})/2$. Now consider the three-dimensional analogy: find the volume of the region of intersection of four spheres of radius r, each of which has its center on the surfaces of the other three.

Only Richard Hess, Emmet Duffy, and Greb Huber submitted solutions to this problem. Hess' solution is as follows:

The centers of the spheres lie on the vertices of a tetrahedron of side r. Let V_1 be the volume of the tetrahedron and V_2 be the volume subtended by the solid angle of the tetrahedron vertex and the surface of the sphere, as shown below. Then the



volume of the desired intersection is

$V = 4V_2 - 3V_1$ (1)

$V_1 = (r^3 \cdot 2^{1/2}) / 12$ (2)

$V_2 = (4r^3/3) \arctan (\tan \pi / 12)^{1.5}$ (3)

Plugging (3) and (2) into (1) gives $V \simeq .38149407r^3$.

OCT 5 A positive-integer solution to the pair of Pythagorean relations $A^2 + B^2 = C^2$ and $(A + 3)^2 + (B + 3)^2 = (C + 4)^2$ is the trivial triple $(A,B,C) = (0,1,1)$. Find two other positive-integer solutions, and determine if there are a finite or infinite number of such solutions.

The following solution is from Charles Sutton: Given the equations

$A^2 + B^2 = C^2$ (1)

$(A + 3)^2 + (B + 3)^2 = (C + 4)^2$ (2)

It is well-known that solutions of (1) are given by

$A = 2mn,\ B = m^2 - n^2,\ C = m^2 + n^2$ (3)

where m and n are integers. Substituting these equations in (2) and simplifying gives $m^2 - 6mn + 7n^2 - 1 = 0$, which when solved for m yields

$m = 3n \pm \sqrt{2n^2 + 1}$ (4)

Clearly $2n^2 + 1$ must be a perfect square. $n = 0$ gives the trivial solution $(A,B,C) = (0,1,1)$, while $n = 2$ gives $(12,5,13)$ and $(36,77,85)$. A TI 58 calculator was programmed to find values of n for which $2n^2 + 1$ is a perfect square, with the result

$n = 0, 2, 12, 70, 408, 2378, 13860, \text{—}$ (5)

This sequence is interesting, since the ratio of successive terms appears to approach a limit rapidly and suggests the well-known Fibonacci sequence, which behaves similarly. Does the sequence (5) likewise satisfy a second-order recurrence relation? Assuming $u_{k+2} = Pu_{k+1} + Qu_k$ and substituting the first four values from the sequence gives $12 = 2P$, $70 = 12P + 2Q$, from which $P = 6$, $Q = -1$, so

$u_{k+2} = 6u_{k+1} - u_k$ (6)

This is also found to be satisfied by the other members of the sequence and so apparently provides an easy method of continuing the sequence.

To get an explicit formula for any term of the sequence, assume $u_k = r^k$ and substitute in (6). Dividing out r^k gives $r^2 = 6r - 1$, from which $r = 3 \pm 2\sqrt{2}$. Since (5) is a linear difference equation, the general solution is

$u_k = C_1(3 + 2\sqrt{2})^k + C_2(3 - 2\sqrt{2})^k$

Substituting $k = 0$, $u_0 = 0$ and then $k = 1$, $u_1 = 2$ gives $0 = C_1 + C_2$ and $2 = C_1(3 + 2\sqrt{2}) + C_2(3 - 2\sqrt{2})$, from which $C_1 = 1/2\sqrt{2}$ and $C_2 = -1/2\sqrt{2}$. Hence

$$u_k = \frac{1}{2\sqrt{2}} \left[(3 + 2\sqrt{2})^k - (3 - 2\sqrt{2})^k \right] \quad (7)$$

To prove that this formula gives *all* terms of the sequence (5), that is, $u_0 = 0$, $u_1 = 2$, $u_2 = 12$, —, we show that $2u_k^2 + 1$ is a perfect square.

$$2u_k^2 + 1 = \frac{1}{4} \left[(3 + 2\sqrt{2})^{2k} - 2 + (3 - 2\sqrt{2})^{2k} \right]$$

$$= \frac{1}{4} \left[(3 + 2\sqrt{2})^{2k} + 2 + (3 - 2\sqrt{2})^{2k} \right]$$

$$= \left[\frac{(3 + 2\sqrt{2})^k + (3 - 2\sqrt{2})^k}{2} \right]^2$$

We can conclude that there exist infinitely many solutions of the simultaneous equations (1) and (2), and that these solutions (A,B,C) can be most easily written down by using (6) to calculate values of $n = u_k$ in the sequence (5), calculating two corresponding values of m from (4), and finally getting A, B and C from (3). A few such values are shown below:

n	m	A	B	C
0	-1	0	1	1
0	1	0	1	1
2	3	12	5	13
2	9	36	77	85
12	19	456	217	505
12	53	1272	2665	2953
70	111	15540	7421	17221
70	309	43260	88736	100381

Also solved by Emmet Duffy, Frank Carbin, John Woolston, Winthrop Leeds, Mary Lindenberg, R. Mullikin, Winslow Hartford, Mark Oshin, John Bobbitt, Avi Ornstein, Matthew Fountain, Richard Hess, and the proposer, Harry Zaremba.

Better Late Than Never

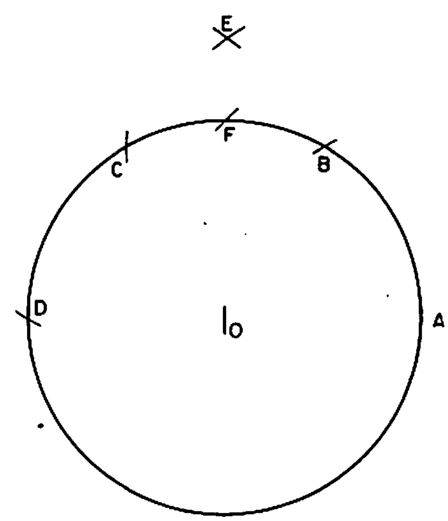
MAY 3 Richard Hess admits that he lost the $26 \cdot 10^{-21}$. As penance he calculated the first 250 digits:

1.2912859970626635404072825905956005414986193682745223173100024451369445387652344555588170411294297089849950709248154305484104874192848641975791635559479136964969741568780207997291779482730090256492305507209666381284670120536857459787030012778941292882 . . .

A/S 3 J. Meier sent a beautiful solution, which he was inspired to write by his knowledge of Professor Norbert Wiener.

Proposers' Solutions to Speed Problems

FEB SD1 Draw the circle with center at 0. Then at any point on the perimeter mark off B, C, and D with chords AB, BC, and CD equal to the radius of the circle.



With a radius equal to AC and centers at A and D, draw arcs intersecting at E. With radius equal to OE and center at A draw an arc intersecting the circle at F. Arc AF is 90°. Proof: Let R equal the radius. Then AC = $\sqrt{3}$R. By construction, AE = $\sqrt{3}$R. If lines were drawn from 0 to A and E and from A to E, triangle OEA would be a right triangle with base R and hypothenuse $\sqrt{3}$R, making OE = $\sqrt{2}$R. Chord AF = $\sqrt{2}$R, which is the length of a chord of a 90°-arc.

FEB SD2 This is, of course, a binary distribution. She received 42 swans and 42 geese, but only 12 drummers and 12 partridges with accompanying pear trees.