Puzzle Corner
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How Many Friends With Birthdays?

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Since this is the first issue of 1981, we are presenting another of our "yearly problems" in which you are to form numbers from the digits 1, 9, 8, and 1 and the arithmetic operators. See the "Problems" section for details and the "Solutions" section for the answer(s) to the 1980 yearly problem.

A short remark on problem backlogs is in order. I have a large supply (greater than a year's worth) of regular, speed, and chess problems; but there is a shortage of bridge problems.

Problems

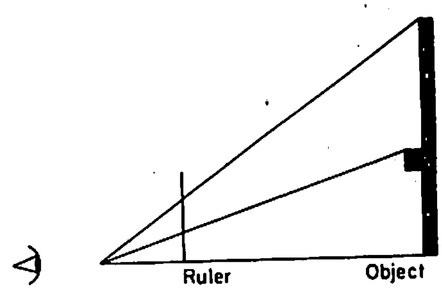
Y 1981 Form as many as possible of the integers from 1 to 100 using the digits 1, 9, 8, and 1 exactly once each and the operators +, −, * (multiply), / (divide), and ** (exponentiation). We desire solutions containing the minimum number of operators; and, for a given number of operators, solutions using the digits in the order 1, 9, 8, and 1 are preferred.

JAN 1 Phillip Feuerwerger wonders what is the probability of picking up a bridge hand containing no suit with exactly three cards.

JAN 2 James Boetler has been unable to find a set of five distinct positive integers such that the sum of each pair is a perfect square. Does such a set exist?

JAN 3 Raymond Cowen has a balancing scale and 15 billiard balls. He knows that 14 of the balls are identical in weight, but the fifteenth is either heavier or lighter. What is the minimum number of balancings needed to isolate the "odd" ball?

JAN 4 Our next problem is from Frank Rubin, who writes:



One good way to estimate the height of an object is to take a known height, sight along a ruler until the known object subtends an easy length to work with, and then take the proportional height subtended by the unknown object. If the ruler is parallel to both objects, the results will be exact. Recently I attempted to measure the height of a bridge this way. I had a friend who was exactly six feet tall stand next to the bridge tower. I held the ruler so that he appeared to be one inch tall; the height of the roadway then appeared to be four inches and that of the supporting tower ten inches. Hence I estimated that the roadway was 24 feet above ground and that the tower was 60 feet high. Later I found that the roadway was actually 26 feet above the ground. Clearly this was because I did not hold the ruler precisely vertical. What, then, was the correct height of the tower?

JAN 5 Our last regular problem, from Ernest Steele, is an extension of the famous birthday problem:

I have occasionally asked an acquaintance to estimate the number of people that would give a 50-percent probability of having two coincident birthdays in one year. My friends are always greatly surprised when I tell them that only 23 are needed. This is determined, of course, by

$1 - [(364)(363)(362) \ldots (366 - n)]/365^{(n-1)}$.

But then I became interested in the problem of the minimum number of people needed to give at least a 50-percent probability that there would be three coincident birthdays in the year.

Speed Department

JAN SD 1 Our first speed problem is from William Katz:

Three equal applicants for promotion are led into a room, blindfolded, seated at a table, and told, "While we were blindfolding you, we put a dark spot on none, one, two, or all three of your heads. When we remove the blindfolds, if you see at least one spot, put your hands on the table. If you *know* what you have, stand up. But if you cannot explain, you lose your chance of promotion. All three were marked. Of course all put their hands on the table when able to see. Finally, B arose. How did he *know*?

JAN SD 2 Blaine Rhoades has a quickie about a quick car:

A car is driven once around a one-mile track at exactly 30 mph. How fast must the second lap be driven in order to average 60 mph for the full two-mile course?

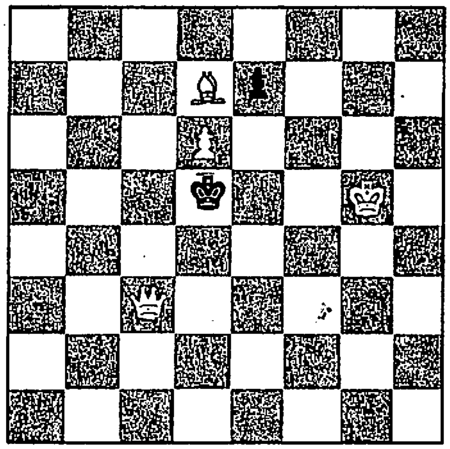
Solutions

Y1980 The problem is the same as Y1981, preceding, except that the numbers 1, 9, 8, and 0 are to be used. As expected, the zero made this a very difficult year. Al Weiss made the best of a bad situation and used leading zeros to reduce the number of operations needed for 11 of the 29 solutions he found. His response follows; he notes that because he was not sure numbers such as "01" were acceptable, he provided alternate solutions in the cases where his best solutions contained such a number.

1:	1**980	11:	91 − 80
2:	018/9	12:	108/9
	1 + 98**0	16:	08 + 9 − 1
4:	8/(1 + 9**0)		0 + 8 + 9 − 1
5:	90/18	17:	18 − 9**0
6:	8 − 1 − 9**0	18:	19 − 8**0
7:	8 + 9 − 10	19:	19 + 8*0
8:	19*0 + 8	20:	180/9
9:	90 − 81	·27:	018 + 9
10:	1**80 + 9		19 + 8 + 0
61:	80 − 19	82:	1*90 − 8
62:	8*9 − 10	83:	091 − 8
63:	(08 − 1)*9)		0 + 91 − 8
	(0 + 8 − 1)*9	88:	98 − 10
64:	(09 − 1)*8	89:	01*89
	(0 + 9 − 1)*8		1*9 + 80
70:	80 − 1 − 9	90:	810/9
71:	1*80 − 9	91:	0*8 + 91
72:	90 − 18	92:	8**0 + 91
73:	01 + 8*9	97:	098 − 1
	1 + 9*8 + 0		0 + 98 − 1
79:	89 − 10	98:	01*98
80:	1**9*80		1*98 + 0
81:	1**9 + 80	99:	19 + 80

Also solved by Jerry Grossman, Winslow Hartford, and Harry Hazard.

A/S 1 The proposer writes that the problem was published incorrectly, with the White King on KR5 instead of KN5. The correct diagram is published below; White is to mate in two; and the problem is now reopened.



A/S 2 Consider a set of N distinct integers, the sum of any K of which is prime. What is the maximum possible value of N for K = 2, 3, 4, and 5?

David Freeman obtained two general results: For K even, the maximum value for N is K. For K prime, $N \le (k-1)^2$. When applied to the cases asked for in this problem, these results yield for 2, 3, 4, and 5:

K = 2, N = 2.
K = 3, N ≤ 4 and (5, 7, 11, 25) works
K = 4, N = 4
K = 5, N ≤ 16.

For this last case a special argument shows that N ≤ 8. Since (19, 49, 79, 121, 151) satisfies the conditions, the maximum N is 6, 7, or 8. Mr. Freeman's analyses are available from the editor.

Matthew Fountain has found that, for K odd, the maximum N ≥ K + 1.

A/S 3 Given four distinct points that lie on (the boundary of) a square, construct the square. Under what conditions is the square uniquely determined?

The following response is from Norman Wickstrand:

By a rare coincidence of events I happened to be looking in Court's *College Geometry* soon after reading "Puzzle Corner," and I found there essentially the same problem. It appears that there are an infinite number of solutions for one unique combination of points. It seems to me that there will be six solutions for all other combinations of points. At least as yet I fail to see how there can be less than six squares. It seems to me that this must have been a very difficult undergraduate challenge.

Responses were also received from Harold Siegel, Elliott Roberts, Harry Zaremba, Emmet Duffy, Naomi Markovitz, and Matthew Fountain.

A/S 4 Find all possible solutions to this cryptarithmetic addition problem:

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THREE
  TWO
  TWO
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SEVEN
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The following solution is from W. Smith: There are two solutions:

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29600
  254
  254
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30108
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E = 0, V = 1, T = 2, S = 3, O = 4, W = 5, R = 6, N = 8, and H = 9.

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79100
  752
  752
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80604
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E = 0, R = 1, O = 2, N = 4, W = 5, V = 6, T = 7, S = 8, and H = 9.
Comment: With nine different letters, base-nine addition would be natural for the problem. I could find no solutions in base-nine arithmetic, however. I did not pursue bases higher than ten. Explanation: I attacked the problem by showing that (i) H must be 8 or 9, and of course S = T + 1; (ii) E must be 0 or 1; and (iii) W must be 0 or 5. Then I examined the cases: (1) H = 8, E = 0, W = 5; (2) H = 9, E = 1, W = 0; (3) H = 9, E = 1, W = 5; and (4) H = 9, E = 0, W = 5. Only case (4) bore fruit. The key to the reasoning is getting insight into what each carry is from one column to the next.

Also solved by Harry Zaremba, Naomi Markovitz, David Freeman, Winthrop Leeds, Matthew Fountain, Gregory Ruffa, Avi Ornstein, Kelly Woods, Frank Carbin, Winslow Hartford, Doug Szper, and the proposer, Theodore Goodman.

A/S 5 Solve for x:
$(3 - \log_a x)/(2\log_a x) + \log_x d = \log_x b + 2\log_x c$.
Express your answer without any parentheses or numerals, using only *one* algebraic operation (addition, multiplication, exponentiation, etc.). You may use that operation as often as you like.

The key is to remember that the log base a of b is the reciprocal of the log base b of a. Then it is easy to solve for x and get

$$x = \frac{a^3d^2}{b^2c^4}$$

Written with just division, this becomes:

$$\frac{\dfrac{\dfrac{a/b}{c/a}}{\dfrac{c/d}{d/c}}}{\dfrac{b}{a/c}}$$

Solutions received from Frank Carbin, Harry Zaremba, Matthew Fountain, Norman Wickstrand, W. Smith, Doug Szper, Winslow Hartford, David Freeman, Everett Leroy, Richard Kruger, Robert Schmidt, and the proposer, Draper Kaufman.

Better Late Than Never

M/A 1 Allen Keith has responded.

M/A 2 Gregory Ruffa solved the N = 8 case under the assumption that at each stage all the squares are occupied.

M/A 4 Ernest Massa has responded.

M/A 5 I. Iverson kindly resubmitted his negative proof, and Emmet Duffy and Harry Zaremba also responded. Iverson's proof follows:
There is such a solid if θ and ϕ as roots are considered as integers, with θ being the Golden Number (0.618034) and ϕ being $\theta + 1$. The dimensions of this solid are θ, 1, and ϕ. Then the face diagonals are $\sqrt{\theta^2+1}$, $\sqrt{3}$, and $\sqrt{\phi+2}$. The space diagonal is 2. But neither θ nor ϕ are integers nor can they be reduced to simple fractions, but since they can represent the desired rectilinear solid, then there cannot be found integers which will do so. The basis of this (negative) proof is that, if such conditions exist for the Golden Number, then it cannot exist for integers. If this should exist for π or for e, the proof would be the same.

J/J 2 Roy Sinclair has responded.

J/J 3 Roy Sinclair, Robert Schmidt, Burton Karpay, Gordon Cochrane, and Marion Weiss have responded.

J/J 5 Burton Karpay, Robert Schmidt, and Gerald Conrad have responded.

NS 15 William Butler submitted an article of Martin Gardner's on this problem. Apparently cyclic solutions (where one player remains stationary and the others sequence along a closed path) are known up to 24 players, and noncyclic solutions can always be found. Mr. Butler's report can be obtained from the editor.

Proposers' Solutions to Speed Problems

SD 1 B says to himself, "I see a spot on A, and C has his hands on the table. If I do not have a spot, A would stand up, seeing me with no spot and C indicating he sees one. Therefore we all have the same problem, and I have a spot."

SD 2 It's not possible.