Puzzle Corner
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Taxes: Five People, Five Calculators

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Doug McIlroy visited us today (27 August) from Bell Labs. After some shop talk about our research on parallel computers, he brought me up to date on BELLE, Ken Thompson's and Joe Gordon's computer chess machine. Their newest version evaluates an astounding one million chess positions per second! This is 100 times faster than any other computerized chess program in existence; extrapolating from these other programs, BELLE is probably playing master-level chess. The world computer chess championships will have been held in Europe by the time you read this, and I am anxious to see how the current champion (Aiken's and Slate's CHESS from Northwestern University) will fare against BELLE. The quality of play will certainly be very high.

William Butler has submitted a revised version of his **FEB 5** problem, which now has five calculators as well as five people. This corrected version appears as **NOV/DEC 3** below.

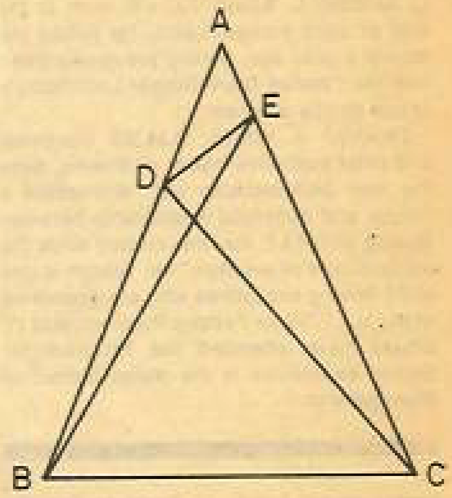
Problems

NOV/DEC 1 Rich Sieger wonders which reachable chess positions require the greatest number of moves to be reached.

NOV/DEC 2 A number-theory problem from Ted Clinkenbeard:
Suppose we are given the sequence
$S0 = \{1^n, 2^n, 3^n, \ldots\}$
and form the difference sequence
$S1 = \{2^n - 1^n, 3^n - 2^n, \ldots\}$
by taking the difference between consecutive elements in S0. Now form S2 as the difference sequence of S1, etc. Show that
$Sn = \{n!, n!, n!, \ldots\}$.

NOV/DEC 3 This is a revised version of **FEB 5**; two new (italicized) sentences have been added:
Five people had consecutive appointments with an income tax expert to help them fill out their 1040 forms and schedules. The electrical engineer had income from a savings account. The man who had a profit trading commodities was taking educational expenses as a deduction. When the man who contributed to a charity was leaving he met the taxpayer with dividend income. The biochemist is deducting interest on a mortgage. The computer programmer uses an SC-40 calculator. *When the man with the medical expense left, he met the man with the educational deduction. The man with four dependents owns a C1400 calculator.* The man with three dependents is claiming storm damage as a deduction. The man with the charitable deduction followed the physicist. The man with five dependents exchanged amenities with the owner of the SR-50. When he looked at the tax expert's calendar, the man with the MX-140 noticed his name was next to that of the man with three dependents. The man with seven dependents sold some real estate for profit. The mathematician has six dependents. The income tax expert still had more than one scheduled appointment after he met the man with dividend income. Each man had a profession, owned a calculator, had a deductible expense, had some number of dependents, and had a second source of income. Who won money in a contest? Who owned an HP-45 calculator?

NOV/DEC 4 Sheldon Katz has a geometry problem for us:



Given angle EBC = 60°, angle DCB = 50°, and AB = AC. Find angle DEB.

NOV/DEC 5 We close the problem department with a cryptarithmetic offering from Frank Rubin:
Replace each letter by a unique decimal digit to obtain a correct multiplication:

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    SINK
    THEM
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DEEPDEEP
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Speed Department

SD 1 A carpentry "quickie" from James Landau:
For a production of *Crucible* the carpentry crew was asked to provide eight wooden stools. The master carpenter took a sheet of plywood 4′ × 8′ and proceeded to cut out eight 16″ × 16″ rectangles on a table saw and then asked me and the other assistant to nail on the legs which had been turned on a lathe the week before. When we said we were finished the master carpenter picked up a 16″ × 16″ piece of plywood and said we had missed one. We just laughed and pointed to the eight completed stools. What happened?

SD 2 A two-part problem from Charles Baker, which reminds me of the first "speed" problem I ever published (in the now-defunct *Tech Engineering News*, where "Puzzle Corner" started):

If a man and a half can dig a hole and a half in a day and half, how long does it take a half a man to dig half a hole? If an acre and a half can grow a foot and a half in a month and a half, how long does it take half an acre to grow half a foot?

Solutions

J/J 1 South's contract was four spades; the opponents made no bid.

North	*South*
♠7 6 3	♠K W 10 4 2
♥A 2	♥K Q 6
♦8 6 5 4	♦A 2
♣A 7 4 3	♣K 9 5

West opened with the ♦K. How does South win?

The following solution is from Robert Slater: South is going to lose the ♠A, one diamond, and one club. His entire problem thus becomes one of avoiding a second trump loser. There is no sense leading out winners: no squeeze will develop, nor can any meaningful discards be made. Therefore, South must assume that a trump distribution exists which will permit making the hand. There are 19 such distributions: any arrangement with three or more spades, including the ♠J, in the West hand will lose; also, ♠A 9 8 5 in the West hand will lose, as will all five trumps in the East hand. The "best chance" play is:

1. Take the opening lead.
2. Return a low heart to the ♥A.
3. Lead a low spade and (unless East goes up with the ♠A) go up with the ♠K or ♠Q.

If that holds (and at the first opportunity, at any event)

4. Return to dummy with ♣A and finesse again, this time playing the ♠10 (unless East lays the ♠J out there).

This approach will win every distribution that can be won except where West has specifically two clubs including the ♣J. It will win any time there are three clubs to the ♣J in East's hand.

Also solved by Doug Van Patter, Winslow Hartford, and the proposer, the late Elmer Ingraham.

J/J 2 Find all integral solutions to $x^2 + y^2 = z^2 + 1$.

Emmet J. Duffy has submitted the following solution:

For even values of x, $x = 2a$, $y = 2a^2 - 1$, $z = 2a^2$, where a is an integer. For odd values of x, $x = 2ab + 1$, $y = ab^2 - a + b$, $z = ab^2 + a + b$, where a and b are integers. These formulas do not give all values of x, y, and z, as the answer $x = 10$, $y = 15$, $z = 18$ cannot be obtained. To get all possible answers, rearrange the equation to:

$x^2 - 1 = z^2 - y^2$

$x^2 - 1 = (z - y)(z + y)$

Let $z - y = F_1$ and $z + y = F_2$. Then $y = (F_2 - F_1)/2$ and $z = (F_2 + F_1)/2$. Thus, take any value of x, square it and subtract 1. Factor the result, remembering that 1 and $x^2 - 1$ are factors. Factors must be both odd integers or both even integers. Half the difference of the factors is y and half the sum of factors is z. A surprisingly large number of prime numbers can be x, y, z. For $x < y < z$, here are some of them:

7	11	13	41	53	67
11	13	17	41	79	89
11	29	31	41	101	109
13	19	23	41	419	421
13	41	43	43	59	73
17	71	73	43	71	83
19	43	47	43	151	157
23	29	37	43	461	463
23	41	47	53	349	353
29	37	47	59	139	151
29	67	73	61	307	313
28	103	107	61	463	467
31	43	53			
31	239	241			

Also solved by Jeffrey Kenton, Robert Lutton, Irving Hopkins, Gerald Blum, Roger Milkman, Mark Lively, Avi Ornstein, Frank Carbin, Raymond Gaillard, Robert Slater, Winslow Hartford, Harry Zaremba, and David Freeman.

J/J 3 An army 40 miles long advances 40 miles while a messenger on horseback rides from the rear of the column to the front and back to the rear. How far has the messenger ridden?

Bob Metcalf did not have much trouble with this problem:

Let M = speed of the messenger, A = speed of the army, U = the messenger's time up to the front of the column, and B = the messenger's time back to the rear.

1. Assuming the army is standing still, the distance up to the front is $(M - A)U = 40$.
2. Assuming the army is standing still, the distance back to the rear is $(M + A)B = 40$.
3. The army travels 40 miles while the messenger goes up and back: $A(U + B) = 40$.
4. The net distance travelled by the messenger up and back is 40, too: $MU - MB = 40$. Adding (1) and (2) we get $M(U + B) = 80 + A(U - B)$. $M(U + B)$ is what we want — the messenger's total distance. Using (3) to substitute for A and using (4) to substitute for $(U - B)$ we get:

$M(U + B) = 80 + 40/(U + B) \cdot 40/M$.

Collecting $M(U + B)$ as X we get:

$X^2 - 80X - 1600 = 0$.

Solving for X we get our answer:

$X = 40(1 + \sqrt{2}) = 96.57$ miles.

Also solved by Emmet Duffy, Jeffrey Kenton, Mark Lively, Gerald Blum, Robert Lutton, Kenneth Kellogg, David Krohn, James Landau, an anonymous self-proclaimed procrastinator, Kenneth Amer, Naomi Markovitz, Avi Ornstein, Everett Leroy, Ken Arbit, Douglas Szper, Ronnie Rybstein, Robert Slater, Peter Sorant, Kelly Woods, Timothy Maloney, David Simen, Winslow Hartford, Liland Jackson, Harry Zaremba, Michael Jung, George Flynn, Mary and Martin Lindenberg, David Freeman, Anne Symanovich, Erik Anderson, Danny Mintz, Charles Swift, and the proposer, John Prussing.

J/J 4 How many one-on-one comparisons would be needed to order by specific gravity six liqueurs if no two have the same specific gravity? Should they be compared individually or in groups?

This problem (when all the circumstances are simplified) asks you to sort six numbers using the minimum number of comparisons. As several readers remarked, the number of comparisons depends somewhat on luck. For example, suppose that you compare A with B finding B greater and then compare B with C finding C greater. With these two comparisons you have determined that $A < B < C$. But if you compare A with B finding B greater and then compare B with C finding B greater, you still must compare A with C. If you are maximally lucky, you can get by with $N - 1$ comparisons — in this case 5. The total number of possible comparisons is $\binom{N}{2}$, in this case 15.

The intent of the problem is to devise a comparison method "independent of luck." This has two common interpretations: a method that minimizes the number of comparisons in the worst case, and a method that minimizes the number of comparisons in the average case. David Freeman solved the worst case problem; no one tried the average case. Mr. Freeman, who needed only ten comparisons (whereas most readers needed 11), was aided by first determining the "best possible" results. Since he achieved these lower bounds, balancing in groups cannot help. Mr. Freeman's solution follows. Since for N items N! orderings are possible, at least k(N) binary comparisons are required to distinguish all cases, where

N! < 2**k(N).

Specifically, we have

N	*k(N)*
1	0
2	1
3	3
4	5
5	7
6	10

We claim that these lower bounds can be achieved for $N < 7$.

If $N = 3$, compare a and b and find, for example, $a < b$ ($b < a$ is similar). Then compare a and c; if $c < a$, $c < a < b$. If $a < c$, compare b and c; if $b < c$, $a < b < c$; if $c < b$, $a < c < b$. Thus we can sort with three comparisons.

If $N = 4$, three comparisons gives $a < b < c$. Then compare d and b; if $b < d$, compare d and c to determine $a < b < c < d$ or $a < b < d < c$. If $d < b$, compare d and a to determine $a < d < b < c$ or $d < a < b < c$. Thus we can sort with five comparisons.

If $N = 5$, we cannot start from the N = case; this is the only nontrivial step. Two compares give $a < b$ and $c < d$. Compare a and c and find, for example, $a < c$ ($c < a$ is similar). Then compare c and e. If $c < e$, compare d and e. If $d < e$, we have $a < c < d < e$ and $a < b$; as with $N = 4$, we can add b to $c < d < e$ with two compares. If $e < d$, reverse labels d and e and proceed as above. If $e < c$, compare a and e. If $a < e$, we have $a < e < c < d$ and $a < b$; as with $N = 4$, we can add b to $e < c < d$ with two compares. If $e < a$, we have $e < a < c < d$ and $a < b$; as with $N = 3$, we can add b to $c < d$ with two compares. In every case, seven compares were used.

If $N = 6$, seven compares gives $a < b < c < d < e$. Now compare c and f and find, for example, $c < f$ ($f < c$ is similar). Compare d and f; if $f < d$, $a < b < c < f < d < e$. If $d < f$, compare e and f; if $e < f$, $a < b < c < d < e < f$; if $f < e$, $a < b < c < d < f < e$. Thus we can sort with ten comparisons.

Since we achieve the minimum with simple weighings, no improvement is possible with other binary strategies, such as comparing in groups.

Also solved by Ken Arbit, Winslow Hartford, Jeffry Kenton, Gerald Blum, Robert Lutton, Naomi Markovitz, Avi Ornstein, Timothy Maloney, and John Prussing.

J/J 5 On a train, Smith, Robinson, and Jones are the fireman, brakeman, and engineer, but not respectively. Also aboard the train are three businessmen who have the same names — a Mr. Smith, a Mr. Robinson, and a Mr. Jones.

1. Mr. Robinson lives in Detroit.
2. The brakeman lives exactly half way between Chicago and Detroit.
3. Mr. Jones earns exactly $20,000 per year.
4. The brakeman's nearest neighbor, one of the passengers, earns exactly three times as much as the brakeman.
5. Smith beats the fireman at billiards.
6. The passenger whose name is the same as the brakeman's lives in Chicago.

Who is the engineer?

William Grosky sent us a nice, neat solution: By 1., Mr. Robinson lives in Detroit. 6. implies that a passenger lives in Chicago; thus, by 2. and 4., the brakeman's nearest neighbor, a passenger, does not live in Chicago or Detroit and is either Mr. Jones or Mr. Smith. By 4. again, either Mr. Jones or Mr. Smith earns exactly three times as much as the brakeman; but $20,000 is not exactly three times any salary. Thus 3. implies that Mr. Smith is the brakeman's nearest neighbor. Using 6., we then have that Mr. Jones lives in Chicago, and thus the brakeman is named Jones. 5. implies that the fireman is Robinson. Thus the engineer is Smith.

Also solved by Winslow Hartford, Avi Ornstein, Gerald Blum, Anne Symanovich, Charles Swift, Peter Sorant, Naomi Markovitz, Robert Lutton, Timothy Maloney, David Freeman, Danny Mintz, Robert Sutton (Robert Sutton is distinct from Robert Lutton), Mike Bercher, Wes Harper, Robert Slater, Thomas Sico, Rich Rosen, Bill Kaiser, Dick Schulze, Al and Jeanne Bloam, Harry Zaremba, Mary and Martin Lindenberg, Michael Jung, David Simen, Kelly Woods, Ronnie Rybstein, Douglas Szper, an anonymous self-proclaimed procrastinator, James Landau, David Krohn, Raymond Gaillard, and Mark Lively.

Better Late Than Never

NS 19 Robert McKean has responded.

FEB 4 Harry Zaremba notes that in the parametric equation for x, the theta term should have a coefficient of 2.

MAY 1, 2, 3, and **4** Richard Hess has responded.

J/J SD 1 Robert Kruger notes the meaning of "meet" allows the boats to meet twice before either has reached the opposite shore.

Proposers' Solutions to Speed Problems

SD1 The master carpenter had taken the 4' × 8' sheet of plywood and cut it three times crossways to give himself three 16" × 48" strips (and a 4' × 4' piece left over) — the only feasible way to handle sheet plywood on a table saw. The first two strips were cut into thirds to give six tops; the third strip also produced three pieces 16" × 16".

SD2 (Courtesy of the editor) A day and a half; a half month.