Puzzle Corner
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When Casey Jones Has a Leaky Boiler

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Since the October issue begins a new volume of *Technology Review*, we take this chance to explain the ground rules of "Puzzle Corner" every year.

In each issue we present five regular problems (the first of which is chess- or bridge-related) and two "speed" problems. Readers are invited to submit solutions to the regular problems, and three issues later one submitted solution is printed for each problem; we also list other readers whose solutions were successful. In particular, solutions to the problems you see below will appear in the February issue. Since I must submit that column sometime in November (today is July 12), you should send your solutions to me during the next few weeks. Late solutions, as well as comments on published solutions, are acknowledged in the section "Better Late Than Never" in subsequent issues.

For "speed" problems the procedure is quite different. Often whimsical, these problems should not be taken too seriously. If the proposer submits a solution with the problem, that solution appears at the end of the same column in which the problem is published. For example, solutions to the October "speed" problems are given below. Only rarely are comments on "speed" problems published or acknowledged.

There is also an annual problem, published in the first issue of each new year; and sometimes we go back into history to republish problems which remained unsolved after their first appearance.

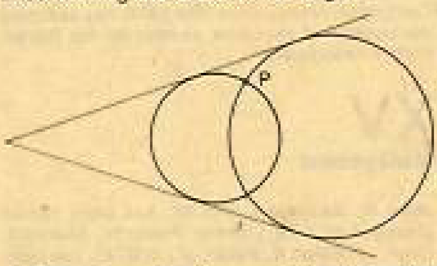
All problems come from readers, and all readers are invited to submit their favorites. I'll report on the size of the backlog, and on the criteria used in selecting problems for publication, in a future issue.

Problems

OCT 1 We begin with a bridge problem from Emmet Duffy. South is on lead with hearts trump and is to take six of the remaining seven tricks against any defense:

	North	
	♠ 8 3 ♥ J 5 ♦ 6 4 2 ♣ —	
♠ J 7 6 ♥ 9 ♦ 9 5 ♣ J		♠ 5 ♥ 6 ♦ Q 10 ♣ K 8 4
	♠ K 4 ♥ — ♦ A J ♣ Q 7 6	

OCT 2 Jon Davis has sent us a geometry problem: An angle and a point within the angle are given. Using only a straightedge and compass, construct the two circles that pass through the point and ar tangent to the lines forming the sides of the angle.



OCT 3 A number theory problem from Neil Hopkins. Consider summing each of the following eight arithmetic progressions:

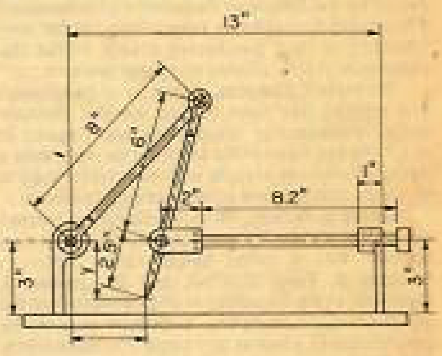
$1 + 2 + 3 + \ldots + n$
$1 + 3 + 5 + \ldots + (2n - 1)$
$1 + 4 + 7 + \ldots + (3n - 2)$
$1 + 5 + 9 + \ldots + (4n - 3)$
$1 + 6 + 11 + \ldots + (5n - 4)$
$1 + 7 + 13 + \ldots + (6n - 5)$
$1 + 8 + 15 + \ldots + (7n - 6)$
$1 + 9 + 17 + \ldots + (8n - 7)$

For each of these series, when $n = 1$ the sum is $1 = 1^2$. Find the one series for which there is no $n > 1$ so that the sum is a perfect square. You might then want to find the two series for which there is an n so that the sum is the perfect number 2305843008139952128.

(A number is called perfect if it equals the sum of its factors — e.g., $28 = 1 + 2 + 4 + 7 + 14$.)

OCT 4 Frank Rubin offers us an energy-related challenge: An old-fashioned steam railroad runs parallel to a river 100 meters away. Unfortunately, the engine boiler has developed a small leak so that there is a loss of one liter of water per second for every ten meters per second the train travels. The conductor is dispatched to refill the train's water reservoir using a ten-liter bucket. He can walk three meters per second with the empty bucket but only two meters per second with the full bucket. What is the fastest constant rate of travel that the train can maintain under these circumstances?

OCT 5 L. Steiger wants to know the values of x and y at the two extreme positions for the device shown below.



Speed Department

OCT SD1 John O'Reilly was looking at Susan Lee's digital clock and noticed that at 12:22:21 the display was symmetric (i.e., 122221 is a palindrome). This led him to ask which two palindromes will occur with the shortest time in between?

OCT SD2 We close with a gambling quicky from William McGuinness. Mike offered Pat the following proposition in a wager on the proverbial beer: "Pick four coins from your pocket at random. Record the last digit in the date of each coin. This will give us four digits. I'll bet that half or more of the digits are the same. If I win, you pay for our next beer. If I lose, I'll pay for it." Who has the better of this proposition?

Solutions

NS17 This problem began as **1978 FEB 3**, which was never completely solved. It was republished as **NS17** in November, 1979. Given an n-by-n checkerboard and n^2 checkers of n different colors, and given that there are n checkers of each color, is it possible to arrange all the n^2 checkers on the board so that no two checkers of the same color lie in the same row, column, or diagonal? (By diagonal is meant all the diagonals, not just the two main diagonals.) After publication of this problem in 1978, an algorithm for placing the checkers was given for n divisible by neither 2 nor 3 (i.e., $n = 1, 5, 7, 11, 13, \ldots$). What remains is to show that no solution is possible for the remaining values of n (or else to find such solutions).

Now Matthew Fountain has submitted the following further partial solution:

The physical basis for my partial solution of **NS17** is that with an even-ordered checkerboard, the board should balance on a center pivot when it contains one checker on each row and column. However, no such balance is possible if one checker is placed on each diagonal parallel to one of the two main diagonals, for all we can do to improve the balance is to change the torque about the main diagonal by switching checkers from one section of a diagonal to the continuation on the other side of the main diagonal. When we compute the torque about the main diagonal for any such starting position, we see that it cannot be reduced to zero, since the torque is not an integer multiple of the change of torque that we can bring about. Proof that no solution exists for checkerboard of even order n: On this board the locus of all squares on one main diagonal is given by $x - y = 0$ where x is the row coordinate and y is the column coordinate. Diagonals parallel to this are of the form $x - y = c - kn$ where c is an integer from 1 to $n - 1$ and k is 0 when $x > y$ and 1 when $x < y$. Let x_i and y_i be the coordinates of the ith checker placed so no two checkers are on the same row, column, or diagonal parallel to $x - y = 0$. If n checkers can be so placed, the following equation is true.

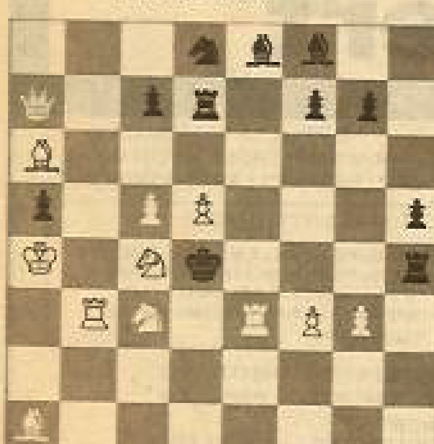
$$\sum_{1}^{n} (x_i - y_i) = \sum_{1}^{n-1} i - mn.$$

Here m is an integer that balances the equation. With one checker in each row and column, the left side is equal to zero. As

$$\sum_{1}^{n-1} i = n(n-1)/2$$

and $(n-1)/2$ is not an integer, the right side can equal zero. Therefore, it is impossible to place n checkers on an even-order board so that only one checker is on each row, column, and diagonal parallel to one given diagonal.

MAY 1 White to play and win:



Many readers found brutal mates in three or four moves, but a quiet move permits a better mate. (By the way, P—B6 is answered by B—B4.) The following joint solution is from John Cronin and John Hornbeck:

White's first move is N—R3. Then . . .

If Black's response is	White's second move is
K × R ch	N—K4 mate
R × P ch	N(B3)—N5 dbl. ch., mate
R—Q3 ch	P—B6 mate
R—K2 ch	P—B6 mate
R—K4	R × R mate
R—R2	R—K4 mate
Any other move	N—B2 mate

Also solved by Jerome Taylor, Robert Sacks, Mike Bobrik, Robert Slater, Emil Friedman, David Krohn, Thomas Sico, Martin Wohl, Norman Wickstrand, G. Gharman, Peter Steven, Steven Feldman, David Simen, and the proposer, John Cronin.

MAY 2 The following two series have sums with a common characteristic for any number of terms n:

$$S_1 = [(m+1)m/2]^2 + (m+m^2)^2 + (m+2m^2)^2 + (m+3m^2)^2 + \ldots$$

$$S_2 = [(m+1)m/2]^2 + (N_2N_3)^2 + (N_3N_4)^2 + (N_4N_5)^2 + \ldots$$

The first term in each series is the sum of the cubes of the initial m positive integers, and the N in S_2 are integers defined as follows: $N_1 = 1; N_2 = m; N_{i+2} = N_{i+1} + N_i$ $(i = 1, 2, 3, \ldots)$ for any positive integers $m = 1, 2, 3, \ldots$. The recursive definition of the numbers N generates Fibonacci numbers when $m = 1$. Find the general expression for the sum of each series and determine the common characteristic of the sums of the series for any m and n.

Only James Lefferts and the proposer, Harry Zaremba, submitted solutions. They agree that the common tract is that both S_1 and S_2 are perfect squares. Specifically

$$S_1 = \left(\frac{1}{2} m(nm+1)(m(n+1)+1)\right)^2 \text{ and}$$

$$S_2 = \left(\frac{1}{2} N_{n+1} N_{n+2} N_{n+3}\right)^2.$$

Mr. Lefferts' solution can be obtained from the editor.

MAY 3 For each $N < 10$, find the shortest possible English word containing n syllables. Do not use proper names, abbreviations, or initials.

Edwin McMillen found some words of Hawaiian origin — namely io, iao, and ieie — which can't be beat for two, three, or four syllables. I have pooled the results of Steve Feldman, Michael Jung, Rich Rosen, and Avi Ornstein to form the following table:

Syllables	Letters	Word(s)
1	1	a
2	2	ai
3	4	area
4	6	acuity
5	7	oxyopia
6	10	ovariotomy
7	12	epidemiology
8	14	diiodobutanoic
9	17	epidemiologically

MAY 4 Queen Bee is a game played by N people $(N > 2)$ who wager a sum of money A in the following manner. Initially, the entire group owes the amount A to an outside party, and each person's expected payout is A/N. Each person flips an unbiased coin, and if there is an "odd man out" that person becomes the "queen bee." If there is no "odd man out" on any particular flip, all players flip again until a "queen bee" is chosen. The "queen bee," when determined, is committed to paying the amount A to the outside party but has the opportunity to win that same amount back from each of the other players by means of a single (unbiased) coin toss by each other player.

1. What is the probability of determining a "queen bee" on each of the group tosses?
2. What is the expected number of tosses required to obtain a "queen bee" for three players? Four players? N players?
3. What is the expected payout for each player, prior to the toss for determination of the "queen bee"?
4. After determination of the "queen bee," what is the net expected gain or loss for each player? (i.e., for "queen bee" and for others).
5. Should the "queen bee" be considered a "winner" or a "loser" at the time he (an inappropriate pronoun — ed.) is chosen?

The following solution is from David Simen:

1. There are 2^N possible configurations for each toss. N of these correspond to there being exactly one head and N to exactly to one tail. Thus the probability of a queen bee being chosen on any given toss is $2N/2^N$.
2. The probability that the queen bee is obtained on the jth toss (and not before) is $(2^N - 2N)^{j-1}/2^N \cdot 2N/2^N$. Therefore, the expected number of tosses required to obtain a queen bee is

$$E = \sum_{j=1}^{\infty} j(2^N - 2N)^{j-1}/2N \cdot 2N/2^N.$$

If we let $p = (2^N - 2N)/2N$, this sum becomes

$$(1 - p)\sum_{j=1}^{\infty} jp^{j-1}.$$

It is well known that this sum is $1/(1 - p)$; a proof follows: let

$$S_k = \sum_{j=1}^{k} jp^{j-1}; \text{ then}$$

$$S_k - pS_k = \sum_{j=1}^{k} p^{j-1} - kp^k = (1 - p^k)/(1 - p) - kp^k,$$

using the formula for the partial sums of a geometric series. Now

$$E = \lim_{k\to\infty} (1 - p)S_k = 1/(1 - p),$$

since $(p < 1)$ $p^k \to 0$ and $kp^k \to 0$ as $k \to \infty$. Thus $E = 1/(1 - p) = 2^N/2N$. For example, if $N = 3$, $E = 4/3$; if $N = 4$, $E = 2$.

3. By symmetry, the expected payout is A/N.
4. Each non-queen bee pays zero with probability ½ and A with probability ½, so his expected payout is A/2. For the queen bee, the probability of winning exactly q tosses $(0 \le q \le N - 1)$, and so taking qA dollars, is

$$\binom{N-1}{q} 1/(2^N - 1).$$

Therefore, the expected gain for the queen bee is

$$-A + \sum_{q=0}^{N-1} \binom{N-1}{q} \cdot qA/2^{N-1}$$

$$= A\left(-1 + 1/2^{N-1} \sum_{q=0}^{N-1} q\binom{N-1}{q}\right).$$

The sum

$$\sum_{q=0}^{N-1} q\binom{N-1}{q} = (N - 1) \cdot 2^{N-2}.$$

This is verified by an easy induction argument. (If you are of little faith, you need two facts to check this:

$$(1)\ \binom{n}{q} = \binom{n-1}{q} + \binom{n-1}{q-1},$$

which is easily checked; and

$$(2)\ \sum_{q=0}^{n} \binom{n}{q} = 2^n,$$

which comes from the binomial expansion of $(1 + 1)^n$.) Therefore the queen bee's expected gain is $A(-1 + (N - 1)/2^{N-1} \cdot 2^{N-2}) = A(N - 3)/2$.

5. Since $A(N - 3)/2 \ge 0$ for $N \ge 3$, while the non-queen bees have negative expected gain, the queen bee is clearly a "winner."

Also solved by Ed Friedman, David Krohn, P. Jung, Harry Zaremba, Frank Carbin, and the proposer, Doug Spizer.

MAY 5 A message switch is a computer hardware/software system that allows messages to come via input lines, stores them, and sends them out on one of many output lines, choosing the time of output and the output line according to programmed criteria. Consider the following simple but large message switch, which has a very large number of input lines and an infinite number of output lines. The output lines are numbered L_1, L_2, etc. Messages come into the system at random, Poisson distributed, with a mean frequency of two per minute. Any message that comes in is immediately output on the lowest-numbered non-busy output line. This immediately makes that output line busy, and it remains busy for one minute, after which it is non-busy until a new message is output on it. The question is: What is the "duty cycle" (percentage of the time the line is busy) of each line L_i, given I? Note that the obvious answer based on the Poisson distribution is wrong. If N messages have been input in the last minute, N output lines will be busy; but they won't necessarily be the first N, because at the time one of these N messages finished coming in all of the first N lines might have been busy with previous messages, forcing this message "up" to a higher-numbered line.

The proposer, Richard King, submitted the following solution:

The duty cycle for L_1 is derived as follows: L_1 receives any message that comes in if it is free, since it is the lowest-numbered line. Its usage consists of a large number of one-minute busy intervals followed by free intervals of varying length. Since the free intervals end with the next arriving message and the latter are Poisson-distributed, averaging two arrivals per minute, the average length of a free interval is half a minute. Thus the duty cycle is 2/3.

Better Late Than Never

NS19 Robert Lutton has responded.
1979 OCT 4 Alan Davis found a substantially simpler solution.

Proposers' Solutions to Speed Problems

OCT SD1 95959 and 100001.
OCT SD2 Pat wins if the four digits are unique. The likelihood of this happening is $10/10 \cdot 9/10 \cdot 8/10 \cdot 7/10 = 63/125$, assuming that the digits are chosen at random. But the digits aren't random, since coins from recent years are more common. This should be enough to change the odds to favor Mike. Try it!