Puzzle Corner
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An X-Rated Problem About Pousse-Cafes

Here are answers to several queries from readers:

Solutions to the "speed" problems appear at the end of the column in the issue in which the problems themselves appear. These problems are not to be taken too seriously, and only in exceptional circumstances do comments concerning "speed" problems appear in the "Better Late Than Never" department.

The backlog of regular problems is large, in excess of two years. For "speed" and chess problems the backlogs are a little smaller — about 18 months. But bridge problems are now in short supply — less than one year backlog.

Problems

J/J 1 Speaking of bridge problems, here is one from the late Elmer Ingraham, who writes that "I've spent too many hours that might well have been better employed to perfect this bridge problem just for your Puzzle Corner in an otherwise quite worthwhile publication":

Repeatedly one of the greatest of bridge players, Mike Gottlieb, once found himself in the position of South with a four-spades contract and looking at these cards after the opponents had made no bid:

North:	South:
♠ 7 6 3	♠ K Q 10 4 2
♥ A 2	♥ K Q 8
♦ 8 6 5 4	♦ A 2
♣ A 7 4 3	♣ K 9 5

West opened with the ♦K. How may South best plan his play to make four spades?

J/J 2 Judith Longyear notes that integral solutions for $x^2 + y^2 = z^2$ are easy to come by. She wants you to find all integral solutions to $x^2 + y^2 = z^2 + 1$.

J/J 3 John Prussing, perhaps anticipating our increased defense research and development budget, submitted a military problem:

An army 40 miles long advances 40 miles while a messenger on horseback rides from the rear of the column to the front and back to the rear. How far has the messenger ridden?

J/J 4 The following problem, from Rudolf Marloth, should not be attempted by anyone under 21 (18 in New York):

While at a spirituous departmental (Hughes Aircraft) get-together one of my epicurean colleagues mentioned that he had started experimenting with pousse-cafés. He used a hydrometer of his own manufacture to determine the order of the layers. My reaction was that only an ordering of specific gravities was needed, not the actual values, so I would have compared equal volumes on a balance. The question is, How many one-on-one comparisons would be needed to order six (never mind N, I'm an engineer) liqueurs if no two have the same specific gravity? Furthermore, could one do better by comparing them in groups?

J/J 5 Our final regular problem was shown to me by John Febbo during a wedding reception a few years ago; it is known as the Smith-Jones-Robinson classic, a "masterpiece of its kind," say the instructions. "It is reported that in one group of 240 people trying it, only six came up with the solution. But there is no 'catch' in it, and the answer has been worked out by many people in five to ten minutes. *Every* fact is important and must be considered." Here it is:

On a train, Smith, Robinson, and Jones are the fireman, brakeman, and engineer, but *not* respectively. Also aboard the train are three businessmen who have the same names — a Mr. Smith, a Mr. Robinson, and a Mr. Jones.

1. Mr. Robinson lives in Detroit.
2. The brakeman lives exactly half way between Chicago and Detroit.
3. Mr. Jones earns exactly $20,000 per year.
4. The brakeman's nearest neighbor, one of the passengers, earns exactly three times as much as the brakeman.
5. Smith beats the fireman at billiards.
6. The passenger whose name is the same as the brakeman's lives in Chicago.

Who is the engineer?

Speed Department

J/J SD 1 A problem from Smith Turner: One boat starts from the east side of a lake and heads west at a constant speed. At the same time another boat starts from the west side and heads east at a constant speed (not necessarily the same as the first boat). When each boat reaches the opposite side it turns around and returns. The boats meet, for the first time, 300 feet from the east side and, for the second time, 400 feet from the west side. How wide is the lake?

J/J SD 2 Ted Mita wants to know: If you are facing a person who is pressing the backspace key on a manual typewriter, in which direction is the carriage moving?

Solutions

FEB 1 White sets up chess pieces in standard form to start a game and has first move. Black sets his king in normal position; he may set what other pieces he uses on any unoccupied squares. For Black to have a forced win,
1. What is the least number of pieces that Black needs, what are they, and where are they placed?
2. If Black is restricted to pawns only, what is the least number needed, and where placed?

For part 1, Edward Friedman notes that if Black has knights on his Q6 and KB6, White is "mated." I use quotes as this position cannot occur in a game, since White must have been in check before Black's last move. Requiring a legal position leads to the following solution from Matthew Chen, James Shearer, Winthrop Leeds and Gardner Perry: N on K5, B on KR5, and Q on KB5. Part 2 is harder. Several readers had solutions with five or six pawns but each had a flaw, usually having White starting by QP × P or KP × P. Matthew Chen, however, has a solution using *nine* pawns placed on QB6, Q6, K6, KB6, KN6, K5, KB5, KN5, KR5. Of course this isn't a legal position. Thus the problem is still open unless someone can show that no solution with eight pawns is possible.

FEB 2 A rope on the roof of a carport, with part hanging over the edge, begins to creep, gains speed, and finally falls entirely to the ground. Suppose the roof was horizontal and perfectly smooth, the rope a slippery, flexible, homogeneous line mass five meters long overhanging one centimeter, and the edge mechanically equivalent to a frictionless sheave of infinitesimal radius, how long would it take the rope to slither off the roof?

The following solution is from Irving Hopkins:

We let:
- L = length of rope = 500 cm.
- x = length of rope hanging down, in centimeters; x_0 = 1 cm.
- g = 980 cm./sec.^{2}
- t = time, in seconds, after release of the rope.
- m = mass of the rope, in g./cm.

The potential energy of rope lost by the 1 centimeter of stationary rope is the weight of it, times the lowering of the center of gravity:
$(m \cdot g \cdot x_0)(x_0/2) = m \cdot g \cdot x_0^2/2$. If the length x is hanging down, the potential energy converted to kinetic energy is
$(mg)(x^2/2 - x_0^2/2)$, with the resulting kinetic energy being $mL(dx/dt)^2/2$. Equating these two and simplifying, we have $g(x^2 - x_0^2) = L(dx/dt)^2$, or
$dx/dt = \sqrt{g/L}\sqrt{x^2 - x_0^2}$

Rearranging this for integration, we have

$$\int_{x_0}^{L} dx/(x^2 - x_0^2)^{1/2} = (g/L)^{1/2} \int_0^t dt, \text{ from which}$$

$\ln\{[L + (L^2 - x_0^2)]^{1/2}/x_0\} = (g/L)^{1/2}t$, which gives t = 4.934 seconds.

Also solved by Peter Wender, Edward Nadler, R. Alward, Everett Leroy, Victor Newton, Winslow Hartford, D. Gupta, Timothy Malony, Anonymous, Michael Jung, Harry Zaremba, John Prussing, Chuck Whitney, Edward Friedman, James Shearer, Winthrop Leeds, Gerald Blum, Gardner Perry, and the proposer, the late R. Robinson Rowe.

FEB 3 In lieu of a conventional key lock, each room in my hotel was equipped with a cipher lock that responded only to the four-digit code selected by the guest when registering. In the course of playing with the lock, one guest noted that the lock would open (indicated by a green LED) whenever the correct code was the last four digits of any sequence. In other words, any amount of garbage could be keyed in; if the last four digits matched, the bolt was energized. It occurred to me that the enterprising burglar would need to try many fewer than the 10,000 possible combinations if he could define a digit stream with the characteristic that each new digit entered resulted in a new four-digit sequence. The minimum number of entries must be 1,003 — four digits to enter the first number, with each of the subsequent 999 resulting in a new sequence. What is the minimum number of keystrokes required?

This problem is part of the computer sciences literature. A summary of an effective algorithm was submitted by Harold Fredricksen and James Maiorana and is reprinted below. Anthony Ralston, a past president of the Association for Computing Machinery, Donald Savage, and J. Swenson also submitted solutions. Timothy Maloney and Gerald Blum noted that 10,003 digits are required.

The solution is a sequence of decimal digits of length 10,003 containing each different length four sequence of decimal digits. In fact it is no restriction to ask that the last three digits of the sequence be identical to the first three digits and then a cycle of length 10,000 is sought. More generally, we require a cycle of length k^n containing every different n-long string of digits chosen from an alphabet of k letters. This problem is known as the Good-deBruijn sequence problem. The cycle asked for in the problem is too long to be presented here, but a smaller example and an exposition of the generating algorithm can be given so that the general n, k cycle can be obtained. For $n = 2$, $k = 4$ we have the cycle 3323130221201100 of length $4^2 = 16$. We obtain the sequence from the algorithm given below. First we define a subtraction operator θ. θ operates on strings of digits of length n to form similar strings, thusly

$a_1a_2 \ldots a_n\theta = (a_1 \ldots a_{j-1}a_j - 1)^q a_1 \ldots a_s$

where $a_j > a_{j+1} = \ldots = a_n = 0$ and $0 \le s < j$. By $(a_1 \ldots a_{j-1}a_j - 1)^q$ we mean that the string $a_1 \ldots a_{j-1}a_j - 1$ is repeated q times and necessarily $n = qj + s$. If $s = 0$, for $a_1 \ldots a_s$ we write the empty string.

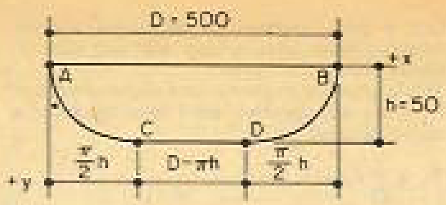
Algorithm (n,k)

1. Initial string is $m = (k - 1)^n$, output $k - 1$.
2. Repeat $m \leftarrow m\theta$ until $s = 0$.
3. Write m as $(a_1 \ldots a_l)^{n/l}$ where l is as small as possible. Output $a_1 \ldots a_l$.
4. If $m = (0)^n$, go to 2, otherwise output $(k - 1)^{n-1}$ and stop.

This last output is not necessary for the length n^k cycle but is necessary for the sequence description of the problem. We use the operator θ instead of the simple arithmetic operation -1 with a test if the resulting n-tuple is the largest element of its equivalence class under cyclic rotation because θ obviously makes the algorithm go faster. A more careful exposition of the algorithm with a proof that it works as well as a guide to related literature can be found in the paper by H. Fredricksen and J. Maiorana in *Discrete Mathematics*, Vol. 23 (1978), pp. 207-210.

FEB 4 It is known that the fastest way to get an object from point a to point b in a uniform gravity field is a cycloid. If a and b are 500 miles apart, the maximum depth of the cycloid would be 159 miles. What is the fastest curve if there is a more severe depth limitation — e.g., 50 miles?

Several readers proposed using part of one cycloid, but I agree with Harry Zaremba *et. al.* that two "half cycloids" connected by a straight line is faster. Mr. Zaremba's solution:



Ignoring friction and air resistance, the fastest path or brachistochrone will be shown in the figure — composed of a half cycloid AC and DE at each end interconnected by a straight tangent line CD. The time for an object to fall a distance h from A to C will be:

$$t = 1/\sqrt{2g} \int_0^{\pi h/2} y^{-\frac{1}{2}}(1 + \dot{y}^2)^{\frac{1}{2}} dx, \qquad (1)$$

in which y is the ordinate to any point, $\dot{y}$ is the slope of the cycloid at the point, and g is the acceleration of gravity (32.2 ft./sec.²). Using the calculus of variations, the parametric equations for the path AC are

$x = h/2 \cdot (\pi - \theta - \sin 2\theta)$

$y = h/2 \cdot (1 + \cos 2\theta)$,

where h is the depth and the parameter θ is the angle between a tangent to the curve and the x-axis. When $\theta = 0$, the horizontal distance travelled from A to C is $x = \pi h/2$. The maximum speed along the line CD will be equal to $v = \sqrt{2gh}$, and it is identical to the velocity attained by an object falling freely through a vertical distance h. This velocity, together with the cycloidal ends, assures an optimum speed and minimum time. Utilizing the parametric equations, integration of equation (1) yields the following time to travel from A to C: $t = \pi\sqrt{h/2g}$. The total time to traverse the path is given by

$T = 2t + (D - \pi h)/v = 2\pi\sqrt{h/2g} + (D - \pi h)/(2gh)^{\frac{1}{2}}$.

For $h = 50$ miles, $T = 841.41$ seconds or 14.02 minutes.

Also solved by John Prussing (who has written a paper on this subject in the *American Journal of Physics* — Vol. 44, No. 3, p. 304), D. Gupta, R. Alward, Winslow Hartford, and the proposer, David Gluss.

FEB 5 Five people had consecutive appointments with an income tax expert to help them fill out their 1040 forms and schedules. The electrical engineer had income from a savings account. The man who had a profit trading commodities was taking educational expenses as a deduction. When the man who contributed to a charity was leaving he met the taxpayer with dividend income. The biochemist is deducting interest on a mortgage. The computer programmer uses an SC-40 calculator. The man with three dependents is claiming storm damage as a deduction. The man with the charitable deduction followed the physicist. The man with five dependents exchanged amenities with the owner of the SR-50. When he looked at the tax expert's calendar, the man with the MX-140 noticed his name was next to that of the man with three dependents. The man with seven dependents sold some real estate for profit. The mathematician has six dependents. The income tax expert still had more than one scheduled appointment after he met the man with dividend income. Each man had a profession, owned a calculator, had a deductible expense, had some number of dependents, and had a second source of income. Who won money in a contest? Who owned an HP-45 calculator?

There seems to be some trouble with this problem as only four calculators were mentioned for five people. I looked at the proposer's *solution* and a C 1400 calculator is included so perhaps part of the problem was inadvertently omitted. Until we hear from Mr. Butler, I offer Roy Blackmer's solution as best fitting the spirit of the problem:

Part of the warm-up for filling out federal tax forms is making educated guesses about items for which there is inadequate documentation. By using my imagination a bit I arrived at the following answers:

The physicist won money in a contest. Incidentally, he has only one dependent (himself) and takes a deduction for alimony. The biochemist owns an HP-45. The mathematician has a TI-12 but does most of his calculating with a PDP-1170. The facts about the five people are:

1. *Physicist*
   - SR 50
   - Deduction for alimony
   - One dependent
   - Won money in contest
2. *Electrical engineer*
   - MX 140
   - Charitable contribution
   - Five dependents
   - Income from savings
3. Computer programmer
   - SC 40
   - Storm damage
   - Three dependents
   - Dividend income
4. Mathematician
   - TI 12
   - Educational deduction
   - Six dependents
   - Commodity profit
5. Biochemist
   - HP-45
   - Mortgage deduction
   - Seven dependents
   - Real estate income

While the above may not qualify as a unique solution, it probably fits the facts more closely than many tax returns.

Responses also received from Winslow Hartford, Yale Zussman, Avi Ornstein, Amnon Stchin, Matthew Fountain, Charles Swift, Matthew Chen, Gardner Perry, Gerald Blum, and the proposer, William Butler.

Better Late Than Never

J/J 1 Mike Bercher and Smith Turner have responded.

A/S 4 Robert Prince and a hard working TI-57 (35 days of computation) have responded.

OCT 2 Irving Hopkins and John Longhaar note that the final formula may be more simply expressed as $x = \cosh(y)$.

OCT 3 John Longhaar has responded, and the following letter (plus a long print-out) was received from Walter Nissen:

This is a curious problem. I have explored many of its nooks and crannies, but I have not found a simple solution. Let

$$N_k = \sum_{i=1}^{k} A_i, \; A_i < A_{i+1}$$

I found a plausible technique while considering $k_x = 3$. It was satisfactory for solving that case ($6 + 19 + 30 = 55$). It also quickly lucks into $k = 4$, where $1 + 22 + 41 + 58 = 122$. Thereafter, I had the patience to write a computer program to do the trial-and-error portion but not to do it by hand. Let $(S_i)^2 = N - A_i$. Suppose that an N not known to be the minimal N_k is found to satisfy all the other requirements, then if $A_i < 1 + 2S_k$, N_k will be associated with the same S_k. This makes the algorithm much more efficient. Results:

$$N_7 = 2210 = 1 + 94 + 185 + 274 + 361 + 529 + 766$$

$$N_8 = 3156 = 20 + 131 + 240 + 347 + 452 + 555 + 656 + 755$$

$$N_9 = 4908 = 8 + 147 + 284 + 419 + 552 + 683 + 812 + 939 + 1064$$

$$N_{10} = 8656 = 7 + 192 + 375 + 556 + 735 + 912 + 1087 + 1260 + 1600 + 1932$$

Note that for $k = 0, 2 \pmod 6$, $S_i = (k-1)^2 + i - 1$ gives rise to a (possibly non-minimal) solution. It is clear that $(x-2)^2 + (x-1)^2 + (x+3)^2 = (x-3)^2 + (x+1)^2 + (x+2)^2$. There are numerous similar relations. With this in mind, perhaps Mr. Duffy would like to give the three partitions of $N_{44} = 12531610$ which satisfy the problem conditions.

Before he mailed that letter, Walter Nissen received the February issue, and he added a postscript pointing out that the solution published there was in error and that he wished his had been mailed in a timely fashion. Then he adds:

I am still quite unsatisfied with my understanding of how to proceed against this problem, but apparently I have picked up a trick or two not recognized by the other solvers. Incidently, my program, which is written in rather inefficient interpretive BASIC, computes cases $k = 7, 8, 9$, and 10 in about six seconds on my Prime 400 minicomputer, although running $k = 2$ thru 210 took many, many hours ($k = 60$ took 6 minutes 44 seconds). Perhaps my use of Algorithm 154 of the Collected Algorithms of the ACM to generate combinations in canonical order and the minimality theorem stated above are responsible for the contrast in efficiency between my program and Al Weiss'. Of course, I am making no claims concerning development time.

NOV 1 Mike Bercher has responded.

NOV 2 Mike Bercher has responded.

NOV 5 Mike Bercher has responded.

D/J 1 Alan LaVergre and Mike Bercher have responded.

D/J 3 Alan Lavergne has responded.

D/J 4 John Longhaar, Alan Lavergne, and A. Alward have responded.

D/J 5 Alan Lavergne and Edward Nadler have responded.

Proposers' Solutions to Speed Problems

SD 1 By the first meeting the boats have combined to make one crossing. By the second meeting, three crossings. Thus the width plus 400 feet that one boat travelled by the second meeting is three times the 300 feet it had gone by the first meeting. So the width is 500 feet.

SD 2 To the left.