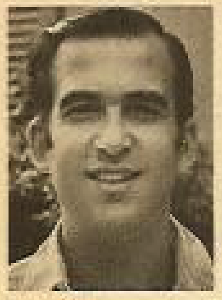
Allan J. Gottlieb

How to Play "Queen Bee"



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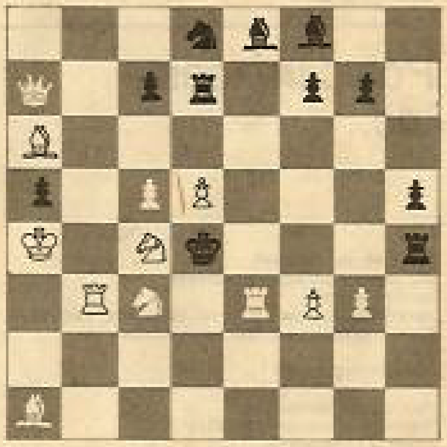
Late in January I fell while running, and 30 stitches were needed to close the wound near my right knee. Fortunately, my wife Alice was with me, brought me to the emergency room, and was very helpful during recuperation; I was fully recovered by mid-February, and now I appreciate the convenience of being able to get around freely. Although I don't recommend using crutches and wearing a Jordan splint, the experience has at least taught me how hard it must be for the physically handicapped. And it is a pleasure to report that many New Yorkers gave me their seats on buses and subways.

Enough of that, and on to the problems.

Problems

NS 19 We begin with a past problem, originally submitted by Richard Orr and published as **1979 D/J 5,** that was never (completely) solved: A game is played by N players where the loser doubles the money of the other N − 1 players. After N games, each player has lost once and each has D dollars. Find d_i, the number of dollars the ith player began with.

MAY 1 Our first new offering is a chess problem from John Cronin: given the situation shown, White to play and win.



MAY 2 Moving from chess to number theory, we present a problem from Harry Zaremba: The following two series have sums with a common characteristic for any number of terms n:

$$S_1 = [(m+1)m/2]^2 + (m+m^2)^3 + (m+2m^2)^3 + (m+3m^2)^3 + \ldots$$

$$S_2 = [(m+1)m/2]^2 + (N_2N_3)^3 + (N_3N_4)^3 + (N_4N_5)^3 + \ldots$$

The first term in each series is the sum of the cubes of the initial m positive integers, and the N in S_2 are integers defined as follows:

$N_1 = 1$; $N_2 = m$; $N_{i+2} = N_{i+1} + N_i$ (i = 1, 2, 3, ...)

for any positive integers m = 1, 2, 3, ... The recursive definition of the numbers N generates Fibonacci numbers when m = 1. Find the general expression for the sum of each series and determine the common characteristic of the sums of the series for any m and n.

MAY 3 Now a word problem from Frank Rubin. For each N < 10, find the shortest possible English word containing n syllables. Do not use proper names, abbreviations, or initials.

MAY 4 Doug Spizer has a game designed for very busy people; he calls it "Queen Bee." It is played by N people (N > 2) who wager a sum A of money in the following manner. Initially the entire group owes the amount A to an outside party, and each person's expected payout is A/N. Each person flips an unbiased coin, and if there is an "odd man out" that person becomes the "queen bee." If there is no "odd man out" on any particular flip, all players flip again until a "queen bee" is chosen.

The "queen bee," when determined, is committed to paying the amount A to the outside party but has the opportunity to win that same amount back from each of the other players by means of a single (unbiased) coin toss by each other player.

1. What is the probability of determining a "queen bee" on each of the group tosses?
2. What is the expected number of tosses required to obtain a "queen bee" for three players? Four players? N players?
3. What is the expected payout for each player, prior to the toss for determination of the "queen bee"?
4. After determination of the "queen bee," what is the net expected gain or loss for each player? (i.e.; for "queen bee" and for others).
5. Should the "queen bee" be considered a "winner" or a "loser" at the time he [an inappropriate pronoun — ed.] is chosen?

MAY 5 Our last regular problem is from Richard King:

A message switch is a computer hardware/software system that allows messages to come via input lines, stores them, and sends them out on one of many output lines, choosing the time of output and the output line according to programmed criteria. Consider the following simple but large message switch, which has a very large number of input lines and an infinite number of output lines. The output lines are numbered L_1, L_2, etc. Messages come into the system at random, Poisson distributed, with a mean frequency of two per minute. Any message that comes in is immediately output on the lowest-numbered non-busy output line. This immediately makes that output line busy, and it remains busy for one minute, after which it is non-busy until a new message is output on it. The question is: What is the "duty cycle" (percentage of the time the line is busy) of each line L_i, given i? Note that the obvious answer based on the Poisson distribution is wrong. If N messages have been input in the last minute, N output lines will be busy; but they won't necessarily be the first N, because at the time one of these N messages finished coming in all of the first N lines might have been busy with previous messages, forcing this message "up" to a higher numbered line.

Speed Dept.

MAY SD 1 Charles Heiberg wants you to find the fallacy in the following argument: Using integration by parts with $u = (\ln x)^{-1}$; $du = -(\ln x)^{-2} \cdot dx/x$; $dv = x^{-1}\,dx$; and $v = \ln x$, one obtains:

$$\int (x \ln x)^{-1}\,dx = 1 + \int (x \ln x)^{-1}\,dx$$

which implies that $0 = 1$.

MAY SD 2 We close with a problem from Emmet Duffy: plot the equation

$\ln [x/(1-x)]$

for various values of x for the range $y = 0$ to $y = 3$. (The x range will be approximately $x = 0.07$ to $x = 0.93$.) Why is the resulting graph:

1. A hell of a curve?
2. Naughty but nautical?
3. American Gothic?

Solutions

J/J 1 As modified in November. The goal is for South, on lead, to win all six remaining tricks;

North: ♠ K 9 / ♥ K / ♦ 4 / ♣ A Q

West: ♠ Q J 10 / ♥ — / ♦ 2 / ♣ J 6

East: ♠ A 8 / ♥ — / ♦ — / ♣ K 5 4 3

South: ♠ — / ♥ A Q J / ♦ 3 / ♣ 9 7

C. Bryant sent us this simple analysis: South leads ♥A, ♥Q, and ♦J, leaving North in the lead with ♠K, ♠9, and ♣A and South holding ♥J, ♣9, and ♣7. If East retains only one spade (the A), North leads ♠9; South ruffs, leading to North's ♣A, and North's ♠K takes the last trick. If East retains ♠A and ♠8, and West only one club, North leads ♣A and then a spade which South ruffs; South then leads clubs for the last trick. If East retains ♠A and ♠8 and West ♣J and ♣6, North leads ♠K; if East covers with ♠A, South ruffs and leads clubs to North's ♣A, whose ♠9 is then good for the last trick; but if East does not cover the ♠K, South discards a club, North then leads ♠9 which South ruffs and leads clubs to North's ♣A for the last trick.

N. Piffenberger used a similar strategy.

D/J 1 The goal is for South, on lead, to win all eight remaining tricks:

North: ♠ A 6 / ♥ K 10 5 2 / ♦ 6 / ♣ 4

West: ♠ K J / ♥ 9 8 7 6 / ♦ 4 / ♣ 2

East: ♠ Q 10 / ♥ K Q 4 3 / ♦ 5 3 / ♣ —

South: ♠ 3 2 / ♥ A / ♦ 2 / ♣ A 10 6 3

After replacing North's ♥K with a ♥J and choosing clubs as trump, Douglas VanPatter continued as follows:

1 — ♥A.

2 — ♣3 to North's ♣4, with East throwing a diamond.

3 — ♥J carried by ♥Q and ruffed with ♣6.

4 — ♣10; West and East each throw a diamond and North throws ♣6.

5 — ♦2 to North's ♦6. West is squeezed; if West throws a heart, East can throw a spade. Now pin the ♥9 by leading ♥10 from North, thus setting up ♥5 in the dummy for the last trick — assuming East covers; otherwise, ♥10 is good. If West throws a spade, East will throw a heart (he cannot throw a spade, since if he does South can cash ♥A, setting up ♣3 in South's hand; now lead ♥5 and ruff out East's ♣K, setting up ♥10 in North's hand with ♣A for reentry.

Responses were also received from Peter Ostapenko, Winslow Hartford, Richard Hess, Ronald Ort, Shirley Wilson, Edwin McMillan, James Prigoff, Harry Hazard, Avi Ornstein, Franklin Seeley, Bo Jansen, N. Piffenberger and Lawrence Felton.

D/J 2 Replace each letter by a different decimal digit to make the arithematic correct:

W H I T E x X = G R E E N
G R E E N x X = B L A C K

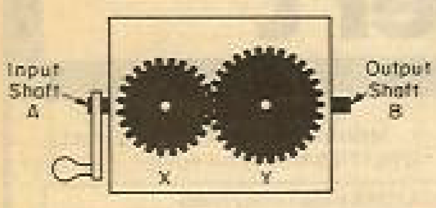
Everyone agrees with the following solutions:

W	H	I	T	E	× X	=	G	R	E	E	N
1	9	0	8	3	× 4	=	7	6	3	3	2
1	5	7	2	3	× 6	=	9	4	3	3	8
G	R	E	E	N	× X	=	B	L	A	C	K
4	8	5	5	3	× 2	=	9	7	1	0	6

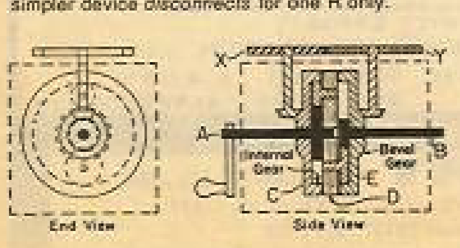
Some readers found alternate solutions with either G or W zero; but leading zeros are not usually allowed in cryptarithmetic problems.

Solutions received from Winthrop Leeds, Emmet Duffy, Avi Ornstein, Dennis Sandow, Steve Feldman, Winslow Hartford, Richard Hess, Naomi Markovitz, and Harry Hazard.

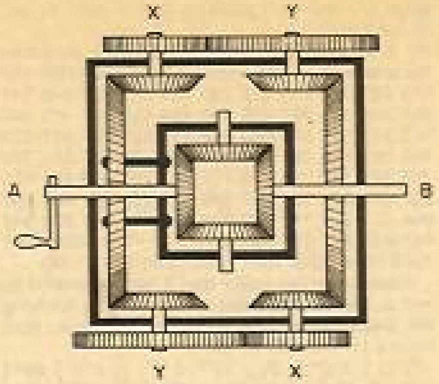
D/J 3 The diagram represents a box from which two keyed shafts project toward the reader. With gears X and Y on these shafts as shown, there is a true drive connection between input shaft A and output shaft B, but if the gears are interchanged shafts A and B can be turned independently. The box contains nothing but ordinary gears, shafts and bearings. What is the mechanism?



Claude von Roesgen suggests that the answer is two differentials. If the transmission ratios are not equal, A and B are independent. He supplied a diagram to illustrate this arrangement; but I am printing instead (below) the original published solution since it is a little clearer and I *believe* incorporates the same idea. (Note: I am *not* a mechanical engineer.) A and B are gear-ended shafts; C and E each carry internal and bevel gears; D carries two planetary idler gears. For a given set of gear ratios "inside" the box, there is *only one* ratio R for X:Y which positively *connects* A and B; a simpler device *disconnects* for one R only.



Winthrop Leeds has a method where the differential box itself rotates:



The back of the box has duplicate X and Y gears as on the front, but they are assembled as shown. All four inside bevel gears are duplicates. The two ring gears are duplicates, but the left one is riveted to the center differential gear box. Gearing locks boxes, so crank A drives shaft B in opposite rotation. If gears X and Y on the front are now interchanged, the differential gear box can rotate and shafts A and B can be turned independently, as required.

Also solved by Raphael Robertazzi, Frank Rubin, James Landau, Edwin McMillan, Richard Merrill, and Richard Hess.

D/J 4 Find the sum of the following series of terms for any positive integer m:

$$S(m) = 1^3 + 2^3(1^3 + 3^3) + 3^3(1^3 + 3^3 + 5^3) + \ldots \ldots + m^3[1^3 + 3^3 + 5^3 + \ldots + (2m-1)^3];$$ or

$$S(m) = \sum_{n=1}^{m} n^3[\sum_{i=1}^{n}(2i-1)^3].$$

Several readers applied Newton's finite difference method to obtain the result directly. Shirley Wilson, however, applied a few summation formulas. Since these can be proved easily by induction (in fact, Dennis Kluk submitted a solution with such a proof), I am reprinting Ms. Wilson's solution instead of one using a finite difference:

It is well known that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \quad (1)$$

(Consult a CRC, prove by mathematical induction, or teach Calculus I for 6 years!)

Multiplying by 8 produces

$$2^3 + 4^3 + 6^3 + \ldots + (2n)^3 = 2n^2(n+1)^2. \quad (2)$$

Replacing n by 2n in (1) gives

$$1^3 + 2^3 + 3^3 + \ldots + (2n)^3 = n^2(2n+1)^2. \quad (3)$$

Thus, from (2) and (3) we have

$$1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = n^2(2n+1)^2 - 2n^2(n+1)^2 = n^2(2n^2-1).$$

$$S(m) = \sum_{n=1}^{m} n^3[\sum_{i=1}^{n}(2i-1)^3] = \sum_{n=1}^{m} n^3[n^2(2n^2-1)] = \sum_{n=1}^{m}(2n^7 - n^5) = 2\sum_{n=1}^{m} n^7 - \sum_{n=1}^{m} n^5.$$

These summations have closed forms listed in a CRC, yielding

$$S(m) = 2\left[\frac{m^2}{24}(m+1)^2(3m^4+6m^3-m^2-4m+2)\right] - \frac{m^2}{12}(m+1)^2(2m^2+2m-1)$$

$$= \frac{m^2}{12}(m+1)^2(3m^4+6m^3-3m^2-6m+3)$$

$$= \frac{m^2}{4}(m+1)^2(m^4+2m^3-m^2-2m+1)$$

$$= \frac{m^2}{4}(m+1)^2(m^2+m-1)^2$$

$$= \left[\frac{m(m+1)(m^2+m-1)}{2}\right]^2$$

Also solved by Frank Rubin, Gerald Blum, George Braun, James Landau, Richard Hess, Emmet Duffy, Winslow Hartford, Frank Carbin, and Irving Hopkins.

D/J 5 The following problem is equivalent to the question of which powers could be computed on a direct-algebraic-notation calculator using parentheses but not numerical keys, memory, or logarithm-based exponentiation keys. (For instance, the seventh power of a number in the x register can be calculated by the sequence: +, x^2, x^2, x^2, x←→y =.) We define the following operations on ordered triples (A,B,C) of positive real numbers.

Mult:	(A,B,C)	⟶	(A×B, O, C)
Div:	(A,B,C)	⟶	(A÷B, O, C)
Load:	(A,B,C)	⟶	(A, A, C)
SQ:	(A,B,C)	⟶	(A^2, B, C)
LP:	(A,B,C)	⟶	(A, A, A)
RP:	(A,B,C)	⟶	(A, C, O)

Say n is admissible if for all z > o, (z^n, o, o) can be obtained from (z, o, o) by fixed (not depending on z) sequence of the above operations. For instance, the algorithm for N = 7 is LOAD, SQ, SQ, SQ, Div. The problem is to find the smallest non-admissible integer.

Richard Hess sent us the following:

M:	(A,B,C)	⟶	(AB, OC)
D:	(A,B,C)	⟶	(A/B, O, C)
LO:	(A,B,C)	⟶	(A, A, C)
S:	(A,B,C)	⟶	(A^2 B, C)
LP:	(A,B,C)	⟶	(A, A, A)
R:	(A,B,C)	⟶	(A, C, O)

(1) Any exponent of A of the form $2^{n_1} \pm 2^{n_2} \pm 2^{n_3}$ is obtainable through applying S n_1 times, then LP, then S $n_2 - n_1$ times, then LD, then S $n_3 - n_2$ times, followed by MRM, DRD, DRM or MRD.
(2) Any exponent of the form $\prod_i (2^{m_i} \pm 2^{m_{i+1}})$ is obtainable through applying S m_i times, then LO, then S $m_j - m_i$ times, followed by M or D.
(3) Any exponent of the form $\prod_i (2^{m_i} \pm 2^{m_{i+1}})$ ± 1 can be obtained by preceding steps (2) with LP and following it with RM or RD.
(4) Any exponent of the form $(2^{n_1} \pm 2^{n_2} \pm 2^{n_3})$ $\prod_i (2^{m_i} \pm 2^{m_{i+1}})$ can be obtained by applying steps (2) after steps (1).
(5) Any exponent of the form $(2^{n_1} \pm 2^{n_2} \pm 2^{n_3})$ $(2^{m_1} \pm 2^{m_2} \pm 2^{m_3})$ can be obtained by applying steps (1) and then applying them again for m_1, m_2, and m_3.
(6) A sieve of Erotosthenes approach produced the following list of inadmissible n up to 1100: 157 173 227 229 233 277 283 313 317 331 346 347 353 367 389 397 439 443 454 457 461 463 467 471 523 547 554 563 569 571 593 607 617 628 643 653 659 661 662 677 683 691 692 694 706 709 727 733 739 773 778 787 794 797 821 823 827 829 831 853 857 859 877 878 886 887 907 908 911 914 916 922 926 932 934 937 939 941 942 947 967 977 983 997 1013 1046 1059 1061 1063 1066 1069 1091 1093 1094 1097 1099

It seems to the editor that if two exponents e_1 and e_2 can be achieved, so can e_1e_2 by applying the steps for e_2 after those for e_1. Thus, from Hess's (1) and (2), we get $\prod_i (2^{m_i} \pm 2^{m_{i+1}}) \prod_{j,k} (2^{n_j} \pm 2^{n_k} \pm 2^{n_l})$. I don't know if this raises the limit above 157. Also solved by P. Jung and Naomi Markovitz.

Better Late Than Never

Y1979 Harry Hazard notes that we used − for + in 8, 17, 21, 22, and 89; and x for − in 74.

A/S 2 James Landau has shown that 86 is the minimum number of days needed even if supply packages can be broken up. In fact, for this situation the number of days needed to cross a 20n + 40 mile desert is

$$2 + \sum_{i=1}^{n} 4^i.$$

He challenges Mr. Bahne to solve the problem for 10n + 40.

PERM 3 Harry Hazard notes that six lines from the bottom of his published solution the 24 should be a 34. George Gerling sends a listing of all four-digit combinations (base 10) and the number of integers each generates in the range 1 to 100, 101 to 135, and 136 to 200, plus the lowest integer not generated; this listing may be obtained from the editor. His results differ somewhat from those reported by Mr. Hazard (*November, 1979, pp. A31-A32*), and on this subject Mr. Gerling observes:

In the cases where I can prove my results correct I have done so by including an example of an integer produced which Mr. Hazard indicates cannot be produced. I cannot prove the opposite — i.e., in cases where he indicates that an integer can be produced and I think he is in error. The most crucial test of this is his statement that the integer 62 can be generated by the combination 1-3-8-9. My program only produced 98 integers for this combination, and if 62 can be generated I have an error in the program. In this case I would no longer conjecture that the non-base-10 solution I submitted is most likely unique. Specifically, as to Mr. Hazard's results:
(1) I agree.
(2) I believe 1-3-8-9 does not generate 62 and 79.
(3) I agree
(4) I agree with 1-3-7-9; but I find that 2-3-7-9 produces 98 integers: 49 = (9 − 2)**3/7.
(5) I agree with his list except for 2-4-5-7, which generates 97 integers.
(6) The combinations 1-2-7-8 lacks 4, not 5, and reaches to 92: 91 = (21 − 8)*7 and 92 = (12*7) + 8.
(7) I show totals for the stated combinations: 1-7-8-9, 76 (he says 77); 2-4-6-8, 78 (he says 77); and 4-5-8-9, 62 (he says 80).

Proposor's Solutions to Speed Problems

MAY SD 1 The integrals on either side of the last equation are indefinite integrals and therefore may differ by a constant.

MAY SD 2 The resulting graph is a trident or pitchfork held by Satan, Neptune, or the farmer in Grant Wood's famous "American Gothic." Is this a well-known curve? Does it have a name?