Puzzle Corner
Allan J. Gottlieb

How the Ancients Moved Their Monoliths?

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Several readers have asked about the deadlines for receiving solutions if they are to escape "Better Late Than Never" treatment. "Puzzle Corner" is written approximately two months prior to your receiving *Technology Review*. For example, today is January 20 (Superbowl Sunday), and I will be mailing my column in a week. This issue is due off the press during the last full week of March. Thus the deadline varies from three to six weeks after you receive *Technology Review*, depending on where the next issue falls in the *Review*'s somewhat irregular publication schedule.

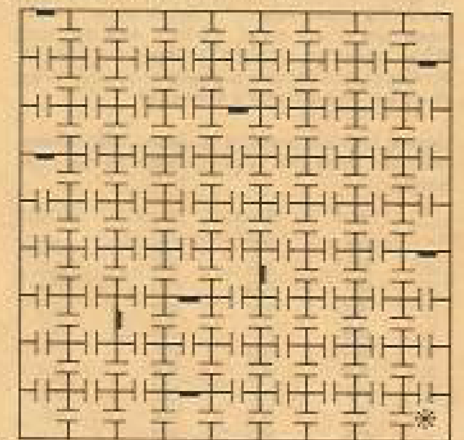
To reduce the number of citations under "Better Late Than Never," I now add names to the "also solved by" list when the galley proofs are reviewed. As a rule, however, the decision as to which solutions to publish cannot be changed at that time.

Jerome Taylor inquired as to the possibility of human computer chess collaborations. This is done occasionally during adjournments, and a "real-time" cooperative effort was tried by David Slate and his co-authored program CHESS 4.9. David Levy beat that team during the recent North American Computer Chess Championship.

Problems

M/A 1 We begin this issue with a bridge problem from Albert Fisher, who wants you to find a hand where declarer's trumps are only A,10 and dummy's only K,7 but in which declarer can make 12 tricks against any opening lead except trump. You are allowed to specify all four hands.

M/A 2 Moving from bridge to prison, we present a problem from Joan Baum, a York College colleague of mine:

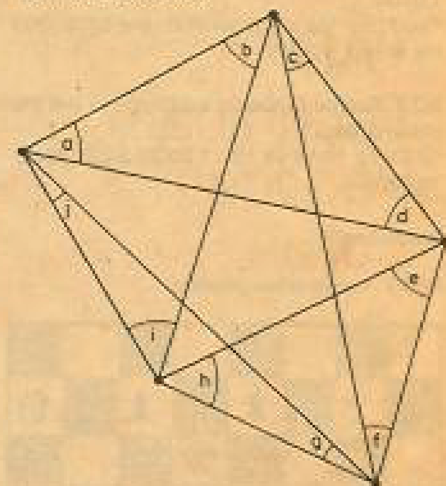


A prisoner was thrown into a medieval dungeon with 145 doors. Nine, shown by black bars, are locked, but each one will open if before you reach it you pass through exactly eight open doors. You don't have to go through every open door, but you do have to go through every cell and all nine locked doors. If you enter a cell or go through a door a second time, the doors clang shut, trapping you.

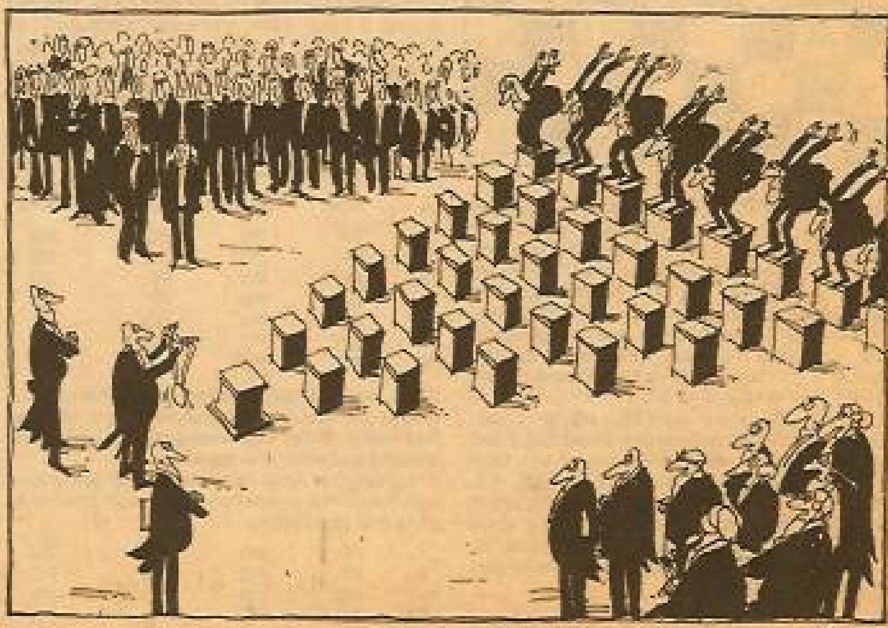
The prisoner (in the lower-right corner cell) had a drawing of the dungeon. He thought a long time before he set out. He went through all the locked doors and escaped through the last, upper-left corner one. What was his route?

M/A 3 The Niculae Asciu cartoon (from the *New York Times, October 9, 1977*) on this page inspired Kenny Goldman to pose a probability problem (direct affirmative-action complaints to Asciu and the *Times*): What is each competitor's chance of reaching the winner's platform? Assume that each interior contestant may jump to one of the two nearest forward squares (50 per cent chance for each) and someone on the end must jump to the nearest forward square. When two contestants land on the same square they flip a fair coin and the loser is eliminated (in the real "Nobel race," the coins aren't always fair. — Ed). If this is too easy, try to find "optional strategies" for jumping right and left (instead of 50-50).

M/A 4 Now let's try a geometry problem from Gary Nelson:



Drawing the diagonals in a convex pentagon results in a five-pointed star. What is the sum of the measure of the angles exterior to



Selecting the Nobel winners (see M/A 3 above). (Niculae Asciu from the New York Times)

the star and bounded by the pentagon. Referring to the diagram, what is:
a-b-c-d-e-f-g-h-i-j?

M/A 5 Problem **1977 O/N 2** asked for three perfect squares the sum of any pair of which is also a perfect square. Since then we have asked for four squares with the sum of any three a square, etc. Richard Hess noted that the original problem is equivalent to finding a rectangular (more precisely rectilinear) solid having integer-length sides and face diagonals. He wonders if one can find a rectilinear solid having integer-length sides, face diagonals, and space diagonal. That is, are there integers A, B, and C such that $A^2 + B^2$, $A^2 + C^2$, $B^2 + C^2$, and $A^2 + B^2 + C^2$ are all perfect squares?

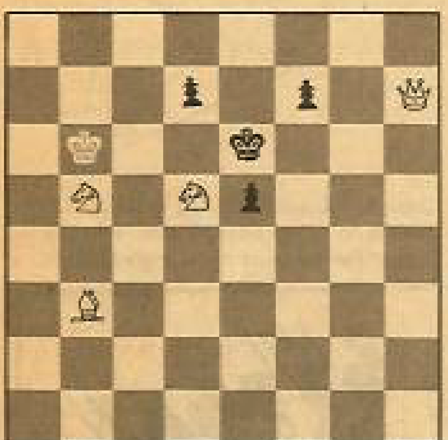
Speed Department

SD 1 Edward Lynch has a secret message for us to decode:
ACBDα LMNOα OSABA R2αGG MNEOK ICD''P HOTI'' RαURI TU1Dα.
Two hints: the α is a goldfish, and try to read the letters quickly.

SD 2 Roland Janberbs wants you to find the missing term:
10 11 12 13 14 15 16 17 20 22 24 31 100 ? 10,000.

Solutions

NOV 1 A mate-in-two problem:



Some readers tried N(N5) — B7 ch, K — Q3, Q — R6. But P — B3 holds out one more move. If this were a solution, the problem would be a poor one since the solution begins with check. I also received K — B7, P — Q3, Q — R3; but again the bishop pawn moves, this time to B4. Smith Turner, G. Sharmon, Richard Hess, and the proposer were not fooled, however. They found the key move: B — R4. Now the five possibilities are: KxN B — N3, P — Q3 N(N5) — B7, P — K5 QxKP, P — B3 N (Q5) — B7, and P — B4 Q — N8. Mr. Turner questioned whether a computer would find short, tricky mates instead of a simple, longer win. Most of the best programs (such as CHESS 4.9, BELLE, and DUTCHESS) use increasing depth exhaustive searches. When they reach three plys (1.5 moves) they examine all variations. So no mate in two is "tricky" to them, and they would not even look at four plys.

NOV 2 The king wanted to find out who was the wisest among his three grand counselors. He blindfolded the three and then announced that he had eight great seals, four purple and four gold. He would place two upon the forehead of each counselor, so that each counselor could see the four seals on his neighbors but not his own two seals. The last two seals he would place in a locked chest. The first counselor who determined the seals in the chest by correct reasoning would become supreme counselor; but if he guessed the answer, either rightly or wrongly, he would be beheaded. The king then put a purple seal and another on the head of Adak, a gold seal and another on the head of Baraz, and two more seals on the head of Cabul. After placing the last two seals in the chest, he removed the blindfolds and began questioning the three counselors. "What seals are in the chest, Adak?" "I do not know." "What seals are in the chest, Baraz?" "I do not know." "What seals are in the chest, Cabul?" "I do not know." Disappointed, the king tried again: "What seals are in the chest, Adak?" "I do not know." "What seals are in the chest, Baraz?" And this time Baraz answered correctly, changed his name to Kissinger, and became supreme counselor. What seals are on each man's forehead?

James Landau presents a clear, step-by-step solution. There are 19 possibilities for the eight seals:

	Adak	Baraz	Cabul	In Chest
1.	PP	PP	GG	GG
2.	PP	GG	PP	GG
3.	PP	GG	GG	PP
4.	GG	PP	PP	GG
5.	GG	PP	GG	PP
6.	GG	GG	PP	PP
7.	PP	PG	PG	GG
8.	PP	PG	GG	PG
9.	PP	GG	PG	PG
10.	PG	PP	PG	GG
11.	PG	PP	GG	PG
12.	GG	PP	PG	PG
13.	PG	PG	PP	GG
14.	PG	GG	PP	PG
15.	GG	PG	PP	PG
16.	PG	PG	GG	PP
17.	PG	GG	PG	PP
18.	GG	PG	PG	PP
19.	PG	PG	PG	PG

When Adak is asked the first time and he doesn't know, he eliminates 3 and 4, because each of these two possibilities has a unique configuration on the foreheads of Baraz and Cabul. Similarly, when Baraz doesn't know when asked the first time, we can eliminate 2 and 5. When Cabul says he doesn't know, we can eliminate 1 and 6. But we can also eliminate both 9 and 12. (There are three possibilities — 2, 3, and 9 — which would have Cabul see Adak wear PP and Baraz wear GG. But 2 and 3 have been previously eliminated, so Cabul eliminates 9. Similarly for 12. On the second run-through, since Adak cannot tell what he is wearing, we can now eliminate 10 (since the only other possibility with PP for Baraz and PG for Cabul is 12, just eliminated by Cabul). By similar reasoning, Adak eliminates 11, 14, and 17. Baraz now has the following possibilities:

	Adak	Baraz	Cabul	Chest
7.	PP	PG	PG	GG
8.	PP	PG	GG	PG
13.	PG	PG	PP	GG
15.	GG	PG	PP	PG
16.	PG	PG	GG	PP
18.	GG	PG	PG	PP
19.	PG	PG	PG	PG

Since each of these seven possibilities has a unique configuration on the foreheads of Adak and Cabul, Baraz now knows exactly what he is wearing and what is in the chest.
It is given that Adak has at least one purple seal and Baraz has at least one gold seal. This limits us to four possibilities:

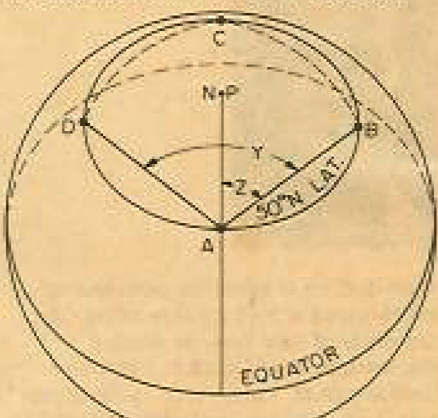
7.	PP	PG	PG	GG
8.	PP	PG	GG	PG
16.	PG	PG	GG	PP
19.	PG	PG	PG	PG

Therefore Bazar is wearing one seal of each color; however, we do not have sufficient information to determine which seals are on Adak or on Cabul or in the chest. It should be mentioned that this problem does *NOT* determine which is the wisest of the three Counselors since 1) each Counselor has to assume the other two are equally good mathematicians 2) the choice of Supreme Counselor was forced by the King when he placed the seals.

Also solved by Matthew Fountain, Winslow Hartford, Richard Hess, Avi Ornstein, Howard Oster, and Ronnie Rybstein.

NOV 3 A spherical square ABCD is drawn on a sphere of radius R and center O, with each arc AB, BC, CD, and DA subtending an angle X at O and each vertex angle of size Y. Find Y as a function of X and the area of the square.

Irving Hopkins was able to navigate all the way through Roy Sinclair's problem. His solution:



Two formulas used in navigation, and one other, are necessary here. The first, (1), is used to calculate the great-circle distance between two points on the earth, corresponding to X in the current problem. The second, (2), finds the azimuth, Z, or the direction of the path at the take-off point, in terms of degrees clockwise from true north. The drawing shows that Y = 2Z. The third comes from the theorem stating that the area of a spherical triangle is equal to the square of the radius times the amount by which the sum of its angles, in radians, exceeds π. Given the latitudes of the points L_1 and L_2 and their longitude difference D,

$$X = \arccos (\cos L_1 \cdot \cos L_2 \cdot \cos D + \sin L_1 \cdot \sin L_2) \quad (1)$$

$$Z = \arccos [(\sin L_2 - \cos X \cdot \sin L_1)/\sin X \cdot \cos L_1] \quad (2)$$

If the North Pole is at the center of the square — an excellent arrangement for our purposes — the corners of the square are all at latitude L and the longitudinal differences D are all 90°. The above equations then become:

$$X = \arccos (\sin^2 L) \quad (1)$$

$$Y = 2Z = 2 \arccos ([\sin L \cdot (1 - \cos X)]/\sin X \cdot \cos L) \quad (2)$$

To find Y as a function of X, as requested,

$$\sin^2 L = \cos X \quad \text{(from 1)}$$

$$\sin L = \sqrt{\cos X}$$

$$\cos L = \sqrt{1 - \sin^2 L} = \sqrt{1 - \cos X}$$

Substituting these expressions in (2), we get

$$Y = 2 \arccos [\sqrt{\cos X} \cdot (1 - \cos X)/\sin X] \quad (3)$$

To find the area, we divide the square by means of a diagonal into two equal triangles. The sum of the angles in each is 2Y, to be expressed in radians. The area of each triangle is $R^2(2Y - \pi)$; the area of the square is twice this, or $A = R^2(4Y - 2\pi)$. Conversely, Y as a function of A is $Y = (A/R^2 - 2\pi)/4$ radians.

Also solved by Richard Hess, Edmond Nadler, Harry Zaremba, and the proposer.

NOV 4 The problem involves a swimmer, his sister, and his uncle. All swim at constant rates of two miles per hour relative to the water; and all want to cross a river one mile wide and flowing at one mile per hour from point A on one side to point C which is one mile downstream on the other side.

1. At what angle should the swimmer point himself, relative to the line AB which is perpendicular to the river? Find the time required and the distance swum.

2. The buoyant sister loves the water but hates exertion. What does she do?

3. The uncle, a self-made man, scorns the nephew's approach (too intellectual) and the niece's (too self-indulgent). He follows his lifelong method of fixing his eye on the objective and steering straight for it. Find his path and how long he takes.

4. Let $V > 2$ be the speed of the river. At what angle θ should the nephew swim to cross the river and land as little as possible downstream from B? And how far downstream will he be?

Harry Zaremba swam and floated his way through this one:

1. Since the time required to traverse the distance AB will equal that required to cover distance BC, we have $t = 1/(2 \cos \theta) = 1/(2 \sin \theta + 1)$, or $2(\cos \theta - \sin \theta) = 1$. Squaring both sides and simplifying, $\sin 2\theta = 3/4$. Hence $\theta = ½ \arcsin 0.75 = 24°17'42.68''$. The nephew's time to reach C will be $t = 1/(2 \cos \theta) = 0.54858$ hours, and the distance swum is $AC = \sqrt{2}$ miles.

2. To minimize exertion, it is presumed the nephew's sister would opt to cross the river in minimum time. If $t = 1/(2 \cos \theta)$ is differentiated with respect to θ and set equal to zero, we get $\sin \theta = 0$. Thus $\theta = 0°$, which is to say that she should swim in a direction perpendicular to the river banks. Her time will be $t = 1/(2 \cos 0°) = 0.5$ hours, and the distance traveled will be $d = (1^2 + 0.5^2)^{½} = 1.118$ miles. To reach point C, she can float downstream the remaining half mile.

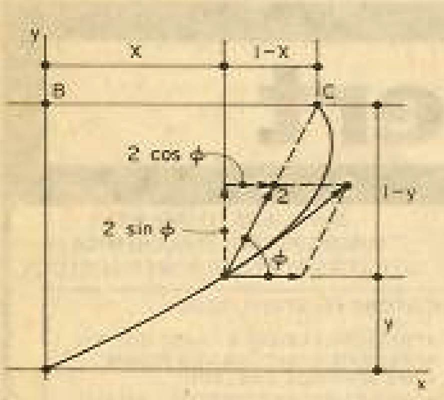
3. At any instant of time, the uncle's swimming component will make an angle ϕ with respect to the river flow. His velocities perpendicular and parallel to the river will be, respectively,

$$dy/dt = 2 \sin \phi \quad (1)$$

$$dx/dt = 2 \cos \phi + 1 \quad (2)$$

in which $\sin \phi = (1 - y)/D$ and $\cos \phi = (1 - x)/D$, with $D = [(1 - x)^2 + (1 - y)^2]^{½}$.
Dividing expression (1) by (2) and eliminating dt,

$$dy/dx = (2 \sin \phi)/(2 \cos \phi - 1)$$
$$= 2(1 - y)/[2(1 - x) + D]. \quad (3)$$



By assuming $w = (1 - x)$, $z = (1 - y)$, and $z = vw$, the variables can be separated in differential equation (3) to permit integration. The following solution of (3) is the uncle's swimming path:

$$(1 - x)^2 = (1 - y)(1 - y - k)^2/4k, \quad (4)$$

in which the constant of integration $k = 3 - 2\sqrt{2}$.
From equation (1), $dt = dy/(2 \sin \theta)$

$$= D dy/2(1 - y). \quad (5)$$

Using a similar approach for integrating (5) as was used for (3), the expression for time at any instant is,

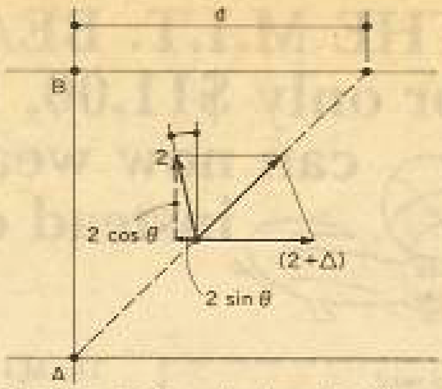
$$t = [(3k + 1) - (1 - y)^{1/2} - 3k(1 - y)^{½}]/6\sqrt{k}. \quad (6)$$

Since $y = 1$ when the uncle reaches point C, his total time to cross the river is derived from (6) to be

$$t = (3k + 1)/6\sqrt{k} = (5 - 3\sqrt{2})/3\sqrt{3 - 2\sqrt{2}} = 0.609475 \text{ hours.}$$

4. Let the river velocity equal $(2 + \Delta)$ where Δ is any excess above 2 m.p.h., and let d equal the distance along the opposite bank downstream of point B. The time elapsed to travel distance d will equal the time required to traverse the width of the river; thus

$t = 1/(2 \cos \theta) = d/[(2 + \Delta) - 2 \sin \theta]$, or
$d = [(2 + \Delta) - 2 \sin \theta]/2 \cos \theta$.



For minimum distance, d is differentiated with respect to θ and set equal to zero. The result is $\sin \theta = 2/(2 + \Delta)$ or $\theta = \arcsin 2/(2 + \Delta)$. Substituting $\sin \theta$ and $\cos \theta = [\Delta(\Delta + 4)]^{½}/(2 + \Delta)$ into the expression for d yields the following minimum distance downstream of B:

$d = [\Delta(\Delta + 4)]^{½}/2$.

NOV 5 Mrs. Black and Mrs. Brown bought cloth, each paying as many cents per yard as she bought yards. Mrs. Green and Mrs. White bought groceries. Mrs. Green spent one cent more than the excess of twice Mrs. Black's payment over seven-ninths of Mrs. Brown's. Mrs. White spent as much as the sum of two-thirds of Mrs. Brown's payment and one-half of Mrs. Green's. Mrs. Black's expenditures was equal to five-sixths of Mrs. Green's payment, plus one-third of Mrs. White's. The total expenditure was more than \$1 and less than \$10,000. How much money, in dollars and cents, did each woman spend?

Winslow Hartford seems delighted ("A diaphantine equation — goody!" he writes) to solve this problem:

Mrs. Black buys x yards of cloth at x cents, spending x^2. Mrs. Brown buys y yards of cloth at y cents, spending y^2. Mrs. Green spends $1 + (2x^2 - 7y^2/9) = v$. Mrs. White spends $2y^2/3 + v/2 = w$. Further, $x^2 = 5v/6 + w/3$. Since v and w must be integers, $y = 3r$ and r must be an odd number. The above equations reduce to:

$$5r^2 + 2x^2 + 1 = 2w$$
$$-7r^2 + 2x^2 + 1 = v$$
$$6x^2 = 5v + 2w.$$

These in turn reduce to $5r^2 - x^2 = 1$. The trivial solution is eliminated because the total amount spent is > \$1. The next solution is $r = 17$, $x = 38$, whence $y = 51$. Mrs. Black spent x^2 = \$14.44; Mrs. Brown spent y^2 = \$26.01; Mrs. Green spent .01 + 28.88 − 20.23 = \$8.66; and Mrs. White spent 17.34 + 4.33 = \$21.67.

Also solved by Ari Ornstein, Frank Carbin, P. Jung, Richard Hess, Harry Zaremba, Naomi Markovitz, Arthur Haines, James Landau, and Steve Feldman. John Rule, the alleged proposer, does not remember submitting this problem.

Better Late Than Never

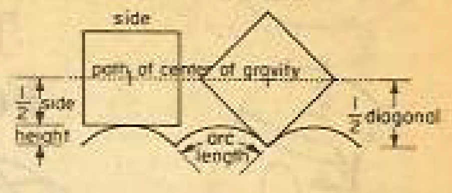
1978 NOV 2 B. Laporte has responded.

1979 A/S 1 Raymond and Edward Gaillard and Robert Enders note that if White plays P—B8(Q), Black draws with R—B5 ch, QxR stalemate. Thus only the P—B8(R) line can be used.

A/S 2 Raymond Gaillard has responded.

OCT 2 James Landau has responded and John Bush, who is this column's proofreader at *Technology Review*, constructed a model (see photo) and has submitted the following:

The expression $y = \log_e (x + \sqrt{x^2 - 1})$ is equivalent to $e^y = x + \sqrt{x^2 - 1}$, and its reciprocal is $e^{-y} = x - \sqrt{x^2 - 1}$. Adding them, we get $e^y + e^{-y} = 2x$, which by definition is equivalent to $x = \cosh y$. This is the equation for a catenary — the curve formed by a chain or flexible string when held up by its ends. In order for the corners of the square to mesh with the cusps in the track, the length of the string, the side of the square, and the arc length of the humps must all be equal. Comparing the square in the two positions, we can see that the height of the hump equals half the diagonal minus half the side of the square:



Therefore, to obtain the correct shape for the hump, take a string as long as the side and hold it up so that its depth equals this height. This technique also works for rectangles and regular polygons; with a little modification it probably works for any polygon. There exist dozens of blocks of stone, weighing hundreds of tons each, at building sites scattered all over the ancient world. Little if anything is known about how the ancient builders moved these blocks. The intriguing possibility is that they discovered this trick with a string without any knowledge of modern mathematics.

OCT 3 The proposer, Emmett Duffy, submitted a solution with this problem.

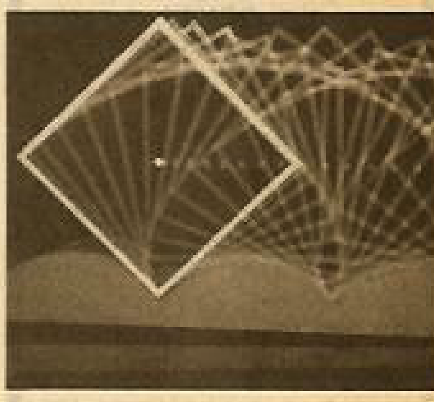
OCT 4 The proposer, John Rule, submitted a solution with this problem; and James Landau has responded.

A/S SD 1 R. Boas, editor of the *American Mathematical Monthly*, writes the following proof for the theorem that there is a correct English sentence containing an arbitrarily large number of consecutive "had"s: Suppose that you already have a correct sentence containing a string of n "had"s. Let us use x to stand for this string, purely as an abbreviation which can be expanded when desired. Now consider the sentence, "My composition would not have contained x had x not been a correct English phrase." This, when expanded, is a correct English sentence containing (2n + 1) consecutive "had"s. (If you replace "contained" by "had," you have (2n + 2), but it's hardly worth the loss in clarity.)

Proposers' Solutions to Speed Problems

SD 1 PH is like Ph in Philadelphia, O is like O in women, and TI is like ti in nation. Hence:
Abie, see the goldfish!
Hell, them ain't no goldfish!
Oh yes, Abie, they are, too, goldfish.
Gee, giminy. Okay, I see the fish are goldfish.
You are right. You won the goldfish.

SD 2 121. The numbers represent 16 written in base 16, base 15, . . ., base 2.



A multiflash photograph of a block of stone rolling down a "scalloped" roadway, a demonstration of John Bush's three-dimensional solution to **OCT 2.** *The road is tipped up a few degrees, and the block moves down it powered only by gravity. The small spot in the middle of the block marks its center of gravity; note that it traces a straight line. (Photo: © 1979 John D. Bush and Glenn Howlett)*