Puzzle Corner
Allan J. Gottlieb

The Annual Anniversary Problem for 1980

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As this is written, I've just returned from the national conference of the Association for Computing Machinery at which the North American Computer Chess Championship was held. Last year's champion, BELLE, designed by Ken Thompson and Joe Condon of Bell Telephone Laboratories, was defeated by CHESS 4.9, the work of Larry Atkin and David Slate of Northwestern University. The top four programs were nearly equal in strength and played very fine chess (I would say they are of "expert" caliber). But a potentially even bigger story is the rapidly growing strength of microcomputer chess programs. Very soon you will be able to purchase chess machines that play high "B" level chess, good enough to beat most amateurs.

Problems

Y1980 This being the first issue for the new year, we begin with our yearly problem: form as many as possible of the integers from 1 to 100 using the digits 1, 9, 8, and 0 exactly once each and the operators +, −, * (multiply), / (divide), and ** (exponentiation). We desire solutions containing the minimum number of operators; and, for a given number of operators, solutions using the digits in the order 1, 9, 8, and 0 are preferred. The solution to Y1979 is given below.

D/J 1 Our first regular problem is from Gary Schwartz; he attributes it to *District 4 Spot*. The goal is for South, on lead, to win all eight remaining tricks.

	♠A 6 ♥K 10 5 2 ♦6 ♣4	
♠K J ♥9 8 7 6 ♦4 ♣2		♠Q 10 ♥K Q 4 3 ♦5 3 ♣—
	♠3 2 ♥A ♦2 ♣A 10 6 3	

[**D/J 1a** As you have discovered by now, the trump suit was not specified. What should it be to make **D/J 1** a good problem? — **Ed.**]

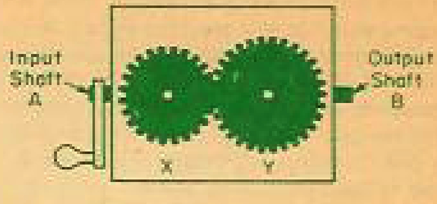
D/J 2 Frank Rubin offers us two cryptarithmetic problems. In each one you are to replace each letter by a different decimal digit to make the arithmetic correct:

W H I T E x X = G R E E N
G R E E N x X = B L A C K

Note that the two problems are *not* related.

D/J 3 The following problem first appeared in *Technology Review* in 1940 as part of an advertisement for Calibron products:

The diagram represents a box from which two keyed shafts project toward the reader. With gears X and Y on these shafts as shown, there is a true drive connection between input shaft A and output shaft B, but if



the gears are interchanged shafts A and B can be turned independently. The box contains nothing but ordinary gears, shafts, and bearings. What is the mechanism?

D/J 4 Harry Zaremba has submitted a problem involving nested series. (I peeked at his solution and can report that it is quite nice.) Find the sum of the following series of terms for any positive integer m:

$$S(m) = 1^3 + 2^3(1^3 + 3^3) + 3^3(1^3 + 3^3 + 5^3) + \ldots + m^3[1^3 + 3^3 + 5^3 + \ldots + (2m-1)^3]; \text{ or}$$

$$S(m) = \sum_{n=1}^{m} n^3 \Big[\sum_{i=1}^{n} (2i-1)^3 \Big].$$

D/J 5 We close with a calculator problem from Alan LeVergne, which he says arose from and is equivalent to the question of which powers could be computed on a direct-algebraic-notation calculator using parentheses but not numerical keys, memory, or logarithm-based exponentiation keys. For instance, the seventh power of a number in the x register can be calculated by the sequence: ÷, x^2, x^2, x^2, x⟷y ÷. We define the following operations on ordered triples (A,B,C) of positive real numbers.

Mult:	(A,B,C)	⟶	(AxB, O, C)
Div:	(A,B,C)	⟶	(A÷B, O, C)
Load:	(A,B,C)	⟶	(A, A, C)
SQ:	(A,B,C)	⟶	(A^2, B, C)
LP:	(A,B,C)	⟶	(A, A, A)
RP:	(A,B,C)	⟶	(A, C, O)

Say n is admissible if for all z > o, (z^n, o, o) can be obtained from (z, o, o) by fixed (not depending on z) sequence of the above operations. For instance, the algorithm for N = 7 is LOAD, SQ, SQ, SQ, Div. The problem is to find the smallest nonadmissible integer. I think the solver will be surprised at how much can be accomplished with such a small set of simple operations; I was.

Speed Dept.

D/J SD1 Ed Lynch has sent us an international problem:

If these sequences of numbers appeared in the following countries, what would they have in common?

Germany	2-10-4-7-6-9-5-1-3-8
France	1-3-6-7-4-9-8-10-2-5
England	2-3-10-6-7-1-9-4-5-8

D/J SD2 Charles Helberg sent us this "speed" problem for people with nondigital watches. Precisely, what is the first time after midnight when the hour and minute hands of the clock coincide?

Solutions

Y1979 Form as many as possible of the integers from 1 to 100 using the digits 1, 9, 7, and 9 exactly once each and the operators +, −, * (multiply), / (divide), and ** (exponentiation). The best solution contains the minimum number of operators and — for a given number of operators — uses the digits in the order 1, 9, 7, and 9.

As one would expect, the duplicate 9s caused this to be a difficult yearly problem. Apparently 33 of the 100 numbers cannot be formed. (Let me remind everyone that a solution with two operators and the digits out of order is preferred over one with three operators and the digits in order.) The following is from Lou Cesa:

1. 1**979	51.
2. (1**9)*9 − 7	52.
3. 19 − 7, − 9	53. 71 − 9 − 9
4. (19 + 9)/7	54. (1 * 9 * 7) − 9
5. (1 + 9)/(9 − 7)	55. (1 − 9) + (7*9)
6. 97 − 91	56. (9 − (1**9))*7
7. (1**99)*7	57.
8. (1**99) − 7	58.
9. (1**97)*9	59.
10. (1**97)+9	60. 79 − 19
11. (1*9) − 7 + 9	61. ((1 + 9)*7) − 9
12. 91 − 79	62. (9*7) − (1**9)
13.	63. (1**9)*7*9
14. (99 − 1)/7	64. (9*9) − 17
15. 9 − (1**9) + 7	65. ((9 − 1)*7) + 9
16. 17 − (9/9)	66.
17. 19 − 7 − 9	67.
18. 17 + (9/9)	68.
19. 1 + ((9 − 7)*9)	69. 79 − 1 − 9
20. (1 + 9) * (9 − 7)	70. (19 − 9)*7
21. 19 − 7 − 9	71. 1 − 9 + 79
22. (91/7) − 9	72. (17 − 9)*9
23.	73. 1 + 9 + (7*9)
24. 9 − 1 + 7 + 9	74. (1*9*9) × 7
25. (1*9) + 7 + 9	75. 91 − 7 − 9
26. 1 + 9 + 7 + 9	76.
27. (1 + 9 − 7)*9	77.
28. 99 − 71	78. 97 − 19
29.	79. (1**9)*79
30.	80. (1**9) + 79
31.	81. (1/9)**(7 − 9)
32.	82. 99 − 17
33.	83. ((1 + 9)*9) − 7
34.	84.
35. 19 + 7 + 9	85.
36.	86.
37.	87. 9 − 1 + 79
38. 19*(9 − 7)	88. 1*9 + 79
39.	89. 1 + 9 + 79
40.	90. ((1**7) + 9)*9
41.	91. 99 − 1 − 7
42.	92. (1*99) − 7
43.	93. 1 + 99 − 7
44. (9*7) − 19	94.
45. (9*(7 − 1)) − 9	95.
46.	96. 97 − (1**9)
47. ((9 − 1)*7) − 9	97. (1**9)*97
48.	98. 19 + 79
49. 7**(1 + (9/9))	99. (1**7)*99
50.	100. (1**7) + 99

Also solved by Phillip Feurwerger, Clark Baker, Tom Schonhoff, Gerald Blum, Ben Ackerman, John Rule, James Bridgeman, Winslow Hartford, Winthrop Leeds, Charles Rivers, G. Sharman, and Harry Hazard.

NS15 There are 4n tennis players who wish to play (4n + 1) doubles matches, where n is any positive integer. How can the matches be arranged so that all players play in every match with the limitation that each player plays with each other player once only and against each other player the same number of times? When n = 1 the solution is easy and quite obvious. Is there a general solution or formula or system? Is it limited to perhaps n = 5 or n = 6?

Judith Longyear attacked this using some methods from the mathematical theory of block designs. Dr. Longyear writes:

Since {a, b, c, d} = ab|cd, ac|bd, ad|bc so nicely, NS15 has a solution whenever 4n = 12t + 4 and there is a resolvable design [(3t + 1)(4t + 1), 12t + 4, 4t + 1, 4, 1]. (Although Dr. Longyear does not tell us what a "resolvable" design is, she does supply examples of pairings for t = 1 and for t = 2—i.e., for 16 and 28 players. I should point out that the latter is our largest example to date and does not fit any of the sequences of values for which solutions were conjectured.)

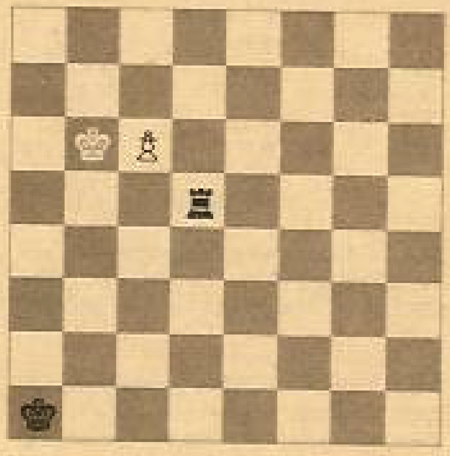
4n = 16. Name the players 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, t, e, w, r, f, and v, and arrange the matches as follows:

Court	Days				
	1,2,3	4,5,6	7,8,9	10,11,12	13,14,15
1	0,1,2,3	0,4,5,6	0,7,8,9	0,t,e,w	0,r,f,v
2	4,9,w,f	1,8,e,f	1,6,w,v	1,5,9,r	1,4,7,t
3	5,7,e,v	2,7,w,r	2,5,t,f	2,4,8,v	2,6,e,9
4	6,8,t,r	3,9,t,v	3,4,e,r	3,6,7,f	3,5,8,w

On days 1, 4, 7, 10, and 13 in each court pair the first and second against the third and fourth. On days 2, 5, 8, 11, and 14 pair the first and third; and on days 3, 6, 9, 12, and 15 pair the first and fourth.

4n = 28. First note that this is the first example found where n is not a power of 2. Name the 28 players $ (a money player!—*Ed.*) and all (x,y,z) where x, y, and z are 0, 1, or 2 and all additions take place modulo (3,3,3). Again, as shown in the table below, three days are spent with each arrangement using the same partnerships as before, where the arrangements run through the nine possibilities for (y,z), namely 00, 01, 02, 10, 11, 12, 20, 21, and 22.

A/S 1 White to play and win:



Let's begin with Robert Slater's solution at the top of the next column:

1	P—B7[1]	R—Q3 ch
2	K—N5[2]	R—Q4 ch
3	K—N4	R—Q5 ch
4	K—N3	K—N1[3]
5	P—B8 (Q)	R—Q6 ch
6	K—B5[4]	

At this point, though many variations are possible due to the mobility of the remaining pieces, White has a clear win.

[1]If K—R7, K—N7, K—B7, or K—R6, loss of pawn is inevitable.
[2]If K—R7, K—N7, or K—R5, loss of pawn is inevitable; if K—B5 and R—Q8, while pawn may be queen, loss is unavoidable.
[3]Not . . . R—Q6 (ch), which will result in queening of the pawn or loss of the rook after K—B2.

If Black actually tries 4 . . . R—Q6 (ch), White can do better than just obtaining a queen-vs.-rook ending, as shown in the following solution by Paul Debbas. Incidentally, the computer program BELLE has shown that queen vs. rook does lead to a win, but *not* easily.

1	P—B7	R—Q3 ch
2	K—N5	R—Q4 ch
3	K—N4	R—Q5 ch
4	K—N3	R—Q6 ch
5	K—B2	R—Q5
6	P—B8 (R)	R—QR5
7	K—N3	

If 7 . . . K—N8, then KxR. Any other move allows R—B1 (mate).

Also solved by Abe Schwartz, Jerome Taylor, Larry Marden, Elliot Roberts, Ron Adelsman, Adam Becker, Coran Li, Turner Gilman, Smith Turner, Jerry Grossman, Don Froemke, G. Sharman, and the proposer, Steve Grant.

A/S 2 An expedition seeking to cross a 100-mile desert can travel 20 miles in a day. However, there are no supplies in the desert, and they can only carry two days' provisions at a time (today's plus one extra day's). They can, of course, go out into the desert, deposit a day's supplies, and return to their base. Supplies are sealed in one-day packages; they cannot be broken up. What is the least number of days it will take to cross the desert? You might also try two variants: Is there a general solution for deserts of different widths? Would the answer be different if the provision against fractional days' supplies were eliminated?

Only the proposer, Charles Bahne, solved this one. His solution is 86 days, using the procedure given in the chart at the top of the next column.

Court				
1	$,	(0,y, z)	(1,y, z)	(2,y, z)
2	(0,1 + y,z)	(0,2 + y,z)	(1,y, 1 + z)	(2,y, z)
3	(1,1 + y,z)	(1,2 + y,z)	(2,y, 1 + z)	(2,y, 2 + z)
4	(2,1 + y,z)	(2,2 + y,z)	(0,y 1 + z)	(0,y, 2 + z)
5	(0,1 + y,2 + z)	(1,1 + y,1 + z)	(0,2 − y, 1 + z)	(1,2 + y,2 + z)
6	(1,1 + y,2 + z)	(2,1 + y,1 + z)	(1,2 − y, 1 + z)	(2,2 + y,2 + z)
7	(2,1 + y,2 + z)	(0,1 + y,1 + z)	(2,2 − y, 1 + z)	(0,2 + y,2 + z)

On day(x) number	Start at mile	Do the following	Return to mile	After which supplies are stored at
1-42	0	Go to mile 10 and store one package there	0	Two at mile 0, 42 at mile 10
43	0	Go to mile 15 and store one package there	10	42 at mile 10, one at mile 15
44-63	10	Go to mile 20 and store one package there	10	Two at mile 10, one at mile 15, and 20 at mile 20
64	10	Go to mile 20 and store one package; then go back to mile 15, pick that one up, and return to	20	22 at mile 22
65-74	20	Go to mile 30 and store one package there	20	Two at mile 20, 10 at mile 30
75	20	Go to mile 35 and store one package there	30	Ten at mile 30, one at mile 35
76-79	30	Go to mile 40 and store one package there	30	Two at mile 30, one at mile 35, and four at mile 40
80	30	Go to mile 40, store one package there, then go back to mile 35, pick that one up, and return to	40	Six at mile 40
81, 82	40	Go to mile 50 and store one package there	40	Two at mile 40, two at mile 50
83	40	Go to mile 55 and store one package there	50	Two at mile 50, one at mile 55
84	50	Go to mile 60, store one package there, then go back to mile 55, pick that one up, and return to	60	Two at mile 60
85	60	Go straight through to mile 80	80	One at mile 80
86	80	Go straight through to	100	None

A/S 3 The following are the first letters of five words of a common, ordered series having to do with numbers. For example, OTTFF is "One, Two, Three, Four, Five." What are the words?

1. U T H T T
2. F S T F F
3. S D T Q Q
4. O F N S T
5. O S T S O

The first four of these are straightforward, and nearly everyone agreed that the answers are:

1. **U**nits, **T**ens, **H**undreds, **T**housands, **T**en **T**housands
2. **F**irst, **S**econd, **T**hird, **F**ourth, **F**ifth
3. **S**ingle, **D**ouble, **T**riple, **Q**uadruple, **Q**uintuple
4. **O**ne, **F**our, **N**ine, **S**ixteen, **T**wenty-five (the squares of the first five whole numbers)

For the fifth, the answers are unique. The reason for this can be found by examining the proposer's (William Hornick) solution:

5. **O**ne, **S**ix, **T**wenty-seven, **S**ixty-four, **O**ne hundred twenty-five (the cubes of the first five whole numbers)

Unfortunately, $2^3 = 8$. But this only slowed some people down momentarily. Avi Ornstein and Harry Zaremba figured out the error; Abe Schwartz hit upon:

5. **O**ne, **S**ixteen, **T**hirty-six, **S**ixteen, **O**ne (the squares of the binomial coefficients 1, 4, 6, 4, 1).

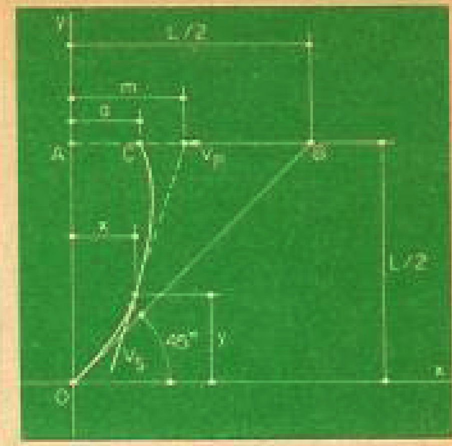
G. Sharman found a complicated method based upon adding and subtracting 6s. Winslow Hartford's solution is even less obvious. But my favorite is from James Landau, who suggests four alternatives:

5. **O**ne, **S**ix hundred twenty-five, **T**en thousand, **S**ix million two hundred fifty thousand, **O**ne hundred million (1^4, 5^4, 10^4, 50^4, 100^4); or
5. **O**ne, **S**ixteen, **T**wo hundred fifty-six, **S**ixty-five thousand five hundred thirty-six, **O**ne million forty-eight thousand five hundred seventy-six (2^0, 2^4, 2^8, 2^{16}, 2^{20}); or
5. **O**perating **S**ystem/360 with **T**ime-**S**haring **O**ption (OS/TSO, the computer system on which I work); or
5. **O**ne, **S**ix, **T**wenty-eight, and **S**o **O**n (a very sloppy list of perfect numbers).

A/S 4 A surface-to-air missile (SAM) tracks its quarry through a heat sensor in the nose. Assume that there is a large square marked off in the sky. A plane enters the square at the southeast corner at 600 m.p.h. and proceeds on a straight course to the northwest corner. At the same moment that the plane enters the square, a SAM enters the southwest corner of the square going at 1200 m.p.h. Instead of "leading" the plane to intercept it more quickly, the SAM points at the plane at all times, thus traveling in an arc as it alters direction to follow the plane. The plane is following a diagonal line. How far up this diagonal line will the plane go before it is intercepted by the SAM?

The following solution is from Harry Zaremba:

In the figure, let points O and B be the corners at which the missile and plane enter the square, respectively. Also, let



v_p = 600 m.p.h., the velocity of the plane.
v_s = 1200 m.p.h., the velocity of the missile.
m = the coordinate of plane at any instant along the diagonal.
L = the length of a diagonal of the square.
a = the coordinate of the point of interception.

From the figure, the slope of the missile's path at any instant of time is

$dy/dx = \dot{y} = (L/2 - y)/(m - x)$. (1)

If s is the distance the missile has traveled in time t, then $t = s/v_s$, and similarly the time for the plane will be $t = (\tfrac{1}{2}L - m)/v_p$. Equating the time intervals, we get

$s/v_s = (\tfrac{1}{2}L - m)/v_p$, or

$m = L/2 - v_p s/v_s = \tfrac{1}{2}(L - s)$.

Substituting m into equation (1) and solving for s,

$s = L - 2x + (2y - L)/\dot{y}$. (2)

Differentiating expression (2) with respect to x and simplifying,

$ds/dx = [(L - 2y)\ddot{y}]/\dot{y}^2$. (3)

Since $ds/dx = (1 + \dot{y}^2)^{\frac{1}{2}}$, then equation (3) equals

$(1 + \dot{y}^2)^{\frac{1}{2}} = (L - 2y)\ddot{y}/\dot{y}^2$. (4)

If we let $\dot{y} = p$, then

$\ddot{y} = dp/dx = dp/dy \cdot dy/dx = p(dp/dy)$.

Substituting $\dot{y}$ and $\ddot{y}$ into (4) and separating variables,

$dp/p(1 + p^2)^{\frac{1}{2}} = dy/(L - 2y)$.

Integrating, and noting that $p = 1$ when $y = O$, the equation of the slope to the missile's path becomes,

$(\sqrt{p^2 + 1} - 1)/p \cdot (L - 2y)^{\frac{1}{2}} = (\sqrt{2} - 1)\sqrt{L}$. (5)

Substituting $p = dy/dx$ into (5), squaring, integrating, and noting that $y = O$ when $x = O$, the solution to (5) becomes

$\sqrt{L - 2y}/3(\sqrt{2} - 1)\sqrt{L} \cdot [y + (4 - 3\sqrt{2})L] = x - (2 - \sqrt{2})L/3$, (6)

which is the equation for the path of the missile. At the point of interception, $x = m = a$ and $y = L/2$. Thus, from equation (6) we get

$a = (2 - \sqrt{2})L/3$.

Referring to the figure, the distance traveled by the plane to the point of interception is

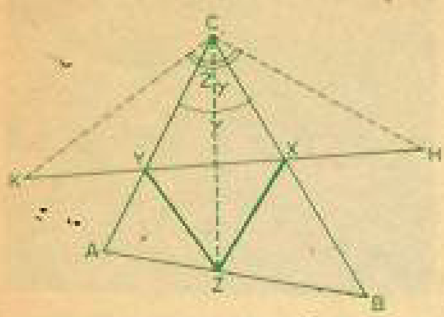
$BC = L/2 - a = L/2 - (2 - \sqrt{2})L/3$, or

$BC = (2\sqrt{2} - 1)L/6 = 0.3047L$.

Also solved by Emmet Duffy, who submitted the published solution to the related hummingbird problem which was published in 1974.

A/S 5 Given an acute-angled triangle, find the inscribed triangle having minimal perimeter.

The following solution is from the proposer, Emmet Duffy: Let the given triangle be *ABC* and let *XYZ* be a triangle inscribed in it, with *X*, *Y*, and *Z* on *BC*, *CA*, and *AB*,



respectively. We will initially consider that *Z* is arbitrarily situated on *AB*; we draw its mirror images *H* and *K* on *BC* and *CA*, respectively, and determine the points of intersection *X* and *Y* of the connecting line *HK* with *BC* and *CA*. For a *fixed* point *Z* the triangle *XYZ* thus formed has the smallest perimeter of all the inscribed triangles. In fact: let *X'* and *Y'* be two other points on *BC* and *CA*. Since *ZX'* and *HX'* are mirror images, and also *ZY'* and *KY'*, and naturally also *ZX* and *HX*, as well as *ZY* and *KY*, the perimeters of the two inscribed triangles to be compared can be written as

$ZXYZ = HX + XY + YK = HK,$

$ZX'Y'Z = HX' + X'Y' + Y'K = HX'Y'K.$

However, since the direct path *HK* from *H* to *K* is shorter than the roundabout path *HX'Y'K*, the first triangle possesses a smaller perimeter than the second. It now merely remains to choose the point *Z* in such manner as to obtain the smallest possible segment *HK* (which represents the perimeter of *XYZ*). Now *CZ* is the mirror image of *CH* and also of *CK*; likewise, $\angle ZCB = \angle HCB$ and $\angle ZCA = \angle KCA$ and thus $\angle HCK = 2\gamma$. Segment *HK* is therefore the base of an isosceles triangle *(HKC)* with a constant apex angle 2γ and the variable leg $s = CZ$; as such it attains a minimum when *CZ* is at a minimum, i.e., when *CZ* is perpendicular to *AB*. Since we could just as easily have carried out the investigation with *X* or *Y* as with *Z*, *AX* is perpendicular to *BC* and *BY* to *CA*. The points *X*, *Y*, *Z* are thus the base points of the altitudes of the triangle *ABC*.

Result: Of all the triangles that can be inscribed in a given acute-angled triangle, the one with the smallest perimeter is the triangle formed by the base points of the altitudes.

Better Late Than Never

1978 A/S 4 Emmet Duffy and an IBM 370 have found three five-digit numbers each satisfying the condition that the sum of the fifth powers of the digits gives the number itself. The numbers are 54,748, 92,727, and 93,084.

1978 D/J 3 Tsvi Ophir has responded.

1979 FEB 1 D. Sweeney notes that the Knight's tour can be solved as a traveling salesman problem. In 1963 he wrote a master's thesis at M.I.T. on this problem which I read a dozen years later when I worked on the same problem.

M/A 3 Tsvi Ophir has responded.

M/A 5 Emmet Duffy notes that the ratio of his implicit solution and the proposer's explicit solution is always near one. Meanwhile, the proposer has withdrawn his explicit solution.

MAY 4 James Landau notes that in the denominator of his continued fraction, "b" should be "2b"; also $b < x < 2b$ should be $b < \sqrt{x} < 2b$. In addition, he enclosed a revised typewritten version of the analysis mentioned in the published solution. Anyone desiring a copy should write to the editor.

J/J 3 Leigh Walzer has responded.

J/J 4 P. Jung has responded.

Proposers' Solutions to Speed Problems

D/J SD 1 The numbers from 1 to 10 written in reverse alphabetical order.

D/J SD 2 From the fact that the hour and minute hands coincide at eleven evenly distributed locations, it follows that the solution is 5-5/11 minutes after 1 p.m.