could not solve this in a few minutes.

Puzzle Corner

Allan J. Gottlieb

Mrs. White's Groceries vs. Mrs. Black's Cloth

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There seems to be some confusion concerning John Rule's challenge presented at the beginning of this department in June/July. We are seeking four digits which — when combined through the operations of addition, subtraction, multiplication, division, and/or exponentiation — yield all the integers from 1 to 100. Following Harry Hazard's suggestion, I am designating this problem as **PERM 3.** Mr. Hazard's preliminary results are presented in the "Better Late Than Never" section, below. For more details, see the description of the "yearly problem" in December/January, 1980.

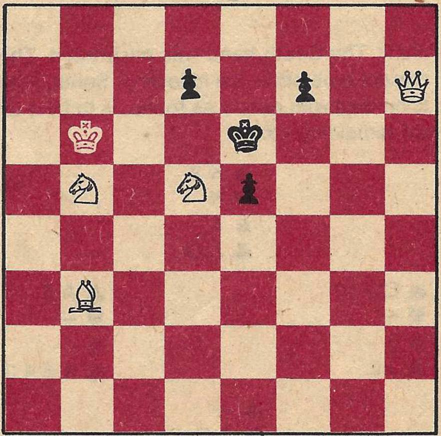
In December/January, nearly a year ago, we presented a problem from Frank Rubin. Unfortunately, we omitted an important phrase. So many readers enjoyed working on this flawed problem that we are reprinting it (correctly, I hope) in this issue.

Problems

NS17 (ne **1978 FEB 3**) We begin with a past problem that was *N*ever (completely) *S*olved, originally submitted by Sheldon Razin: Given an n-by-n checkerboard and n^2 checkers of n different colors, and given that there are n checkers of each color, is it possible to arrange all the n^2 checkers on the board so that no two checkers of the same color lie in the same row, column, or diagonal? (By diagonal is meant *all* the diagonals, not just the two main diagonals.)

When the problem appeared last year, several responses were received. An algorithm for placing the checkers was given for n divisible by neither 2 nor 3 (i.e., n = 1, 5, 7, 11, 13, . . .). Although this algorithm was not proved to generate a desired placement of the checkers, this is not hard to do. What remains to show is that no solution is possible for the remaining values of n (or else find such solutions).

NOV 1 Winthrop Leeds has a mate-in-two for us to try:



By the way, these are exactly the kinds of problems that current computer chess programs handle very proficiently. As this is written, I'm looking forward to attending the North American championships in October, and I'll be surprised if any of the contestants could not solve this in a few minutes.

NOV 2 Here is the Rubin problem mentioned above: The King wanted to find out who was the wisest among his three Grand Counselors. He blindfolded the three and then announced that he had eight Great Seals, four purple and four gold. He would place two upon the forehead of each Counselor, so that each Counselor could see the four seals on his neighbors but not his own two seals. The last two seals he would place in a locked chest. The first Counselor who determined the seals in the chest by correct reasoning would become Supreme Counselor; but if he guessed the answer, either rightly or wrongly, he would be beheaded. The King then put a purple seal and another on the head of Adak, a gold seal and another on the head of Baraz, and two more seals on the head of Cabul. After placing the last two seals in the chest, he removed the blindfolds and began questioning the three Counselors. "What seals are in the chest, Adak?" "I do not know." "What seals are in the chest, Baraz?" "I do not know." "What seals are in the chest, Cabul?" "I do not know." Disappointed, the King tried again: "What seals are in the chest, Adak?" "I do not know." "What seals are in the chest, Baraz?" And this time Baraz answered correctly, changed his name to Kissinger, and became Supreme Counselor. What seals are on each man's forehead?

NOV 3 Roy Sinclair has a spherical geometry problem for us: A spherical square ABCD is drawn on a sphere of radius R and center 0, with each arc AB, BC, CD, and DA subtending an angle X at 0 and each vertex angle of size Y. Find Y as a function of X and the area of the square.

NOV 4 Irving Hopkins submitted a spin-off from **1977 JAN 4.** The 1977 problem, from Ted Mita, was: A swimmer who swims at a constant rate of two miles per hour relative to the water wants to swim directly from point A to a point C, which is one mile downstream and on the other side of a river one mile wide and flowing one mile per hour. At what angle should he point himself, relative to the line AB which is perpendicular to the river? Mr. Hopkins' version is as follows:

1. Prove that the answer to **JAN 4** is (arcsin 0.75)/2, and find the time required and the distance swum (needed for comparison with 2. and 3., below).
2. The swimmer's sister and their uncle also swim at 2 miles per hour. The buoyant sister loves the water but hates exertion. What does she do?
3. The uncle, a self-made man, scorns the nephew's approach (too intellectual) and the niece's (too self-indulgent). He follows his life-long method of fixing his eye on the objective and steering straight for it. Find his path and how long he takes.
4. Let $V > 2$ be the speed of the river. At what angle θ should the nephew swim to cross the river and land as little as possible downstream from B? And how far down-

stream will he be?

NOV 5 Our final regular problem is from John Rule: Mrs. Black and Mrs. Brown bought cloth, each paying as many cents per yard as she bought yards. Mrs. Green and Mrs. White bought groceries. Mrs. Green spent one cent more than the excess of twice Mrs. Black's payment over seven-ninths of Mrs. Brown's. Mrs. White spent as much as the sum of two-thirds of Mrs. Brown's payment and one-half of Mrs. Green's. Mrs. Black's expenditure was equal to five-sixths of Mrs. Green's payment, plus one-third of Mrs. White's. The total expenditure was more than $1 and less than $10,000. How much money, in dollars and cents, did each woman spend?

Speed Department

NOV SD 1 I bet you didn't know that $-1 = 1$. Jukka Korpela sent the following "proof" to the *Sigsam Bulletin:*

$$-1 = i^2 = (i)(i) = \sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1.$$

What is wrong?

NOV SD 2 Another apparent contradiction — this one from Smith Turner: In two contiguous states, the per capita incomes are:

State A — $14,000
State B — $10,000

Jones, living in A, has his pay cut from $13,000 to $12,000 and, feeling that he can no longer afford to live in A, moves to B. Now taking a $13,000 income out of A obviously raises the average there, and putting one of $12,000 into B raises its average. Thus Jones has raised the per capita income in each of two states! But, taking A and B together, all that has happened is that income has dropped by $1,000 so the average of the whole area has dropped. What's wrong here?

Solutions

J/J 1 The South hand was misprinted. The correct problem is the following: South, who is on lead with hearts as trump, is to take all six remaining tricks:

	♠ K 9 ♥ K ♦ 4 ♣ A Q	
♠ Q J 10 ♥ — ♦ 2 ♣ J 6		♠ A 8 ♥ — ♦ — ♣ K 5 4 3
	♠ — ♥ A Q J ♦ 3 ♣ 9 7	

Solutions will be given with those to the November problems. The error was noted by Michael Kotch, Winslow Hartford, Emmet Duffy, William Katz, and Harry Hazard.

J/J 2 A dog crossing a river swims directly towards his master at two miles per hour. The master is directly across the stream at the start. When the dog is two-thirds of the way across the stream his upstream velocity component equals the velocity of the river. The dog swims five minutes longer than if the water had been still. How wide is the river and what is the velocity of the water in it?

The following solution is from W. Woods: The river is one-half mile wide, and the velocity of the water is one mile per hour. The derivation: Let

L = width of river in miles;
v = dog's velocity = 2 miles an hour;
V = velocity of river;
y = downstream deviation from a straight line normal to the starting point; and
x = distance normal to river flow the dog is from his master's side of the river.

Then $y = 0$ when $x = 0$ or $x = L$; and $x = L$ when time $(t) = 0$.

The rate of travel in the x direction is

$$dx/dt = -vx/\sqrt{x^2 + y^2} \qquad (1)$$

The rate of travel in the y direction is

$$dy/dt = V - vy/\sqrt{x^2 + y^2} \qquad (2)$$

Dividing (2) by (1) to eliminate dt, and integrating yields

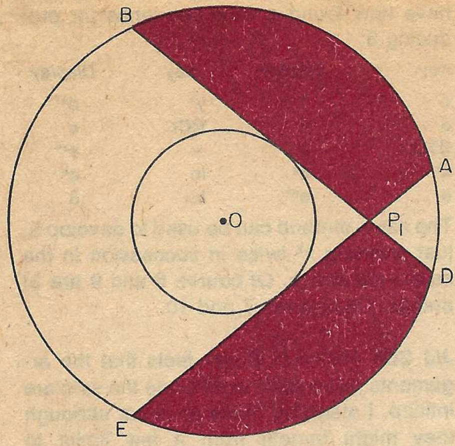
$$y + \sqrt{y^2 + x^2} = x(x/L)^{-V/v} \qquad (3)$$

When $x = L/3$, $dy/dt = 0$. Substituting these values in (2) and (3) yields $V/v = 0.5$. Hence $V = 0.5 \cdot 2 = 1$ mile per hour. Using (3) to define y, integrate (1) from $x = L$ to $x = 0$, obtaining $t = 2L/3$. When $V = 0$, $t = L/v = L/2$. Hence $\Delta t = L/6 = 5/60$ hours; $L = 0.5$ mile.

Also solved by Irving Hopkins, Donald Savage, Conrad Benulis, Mary Lindenberg, and the proposer, Norman Wickstrand.

J/J 3 The inner of two concentric circles has a radius of one inch while the outer has a radius of two inches. A random chord is drawn within the outer circle by the following method: a point (P_1) is located a random distance (0 to 2 inches) at some random direction from the center. A second point (P_2) is determined by the same method. The chord is drawn through the two points. What is the probability that it will intersect the inner circle?

Several readers made the same mistake on this problem: they assumed that probability was proportional to area. That is, since the large circle has four times the area of the small, these readers felt that P_1 has a probability of .25 of being inside the small circle. In fact the probability is .5 since half the possible radii are less than 1 and the radius alone determines whether P_1 is inside the small circle.
Thus, half the time P_1 is inside the small circle, guaranteeing that the chord meets the small circle. For the other half of the time, the proposer, William Butler considers the diagram at the top of the next page.



In order for there to be no intersection P_2 must be in the shaded region. Mr. Butler notes that this probability is twice the probability that P_2 is in B_1A. This later probability is the difference of the probabilities that P_2 is in BDA and that P_2 is in AP_1D. The first of these two probabilities is independent of the location of P_1 whereas the second depends on the radius of P_1. Thus the probability of no intersection is

(1/2) (2) Prob (P_2 in BDA) — Prob (P_2 in AP_1D).

Now some careful arguments (using integral calculus) are needed to calculate the two remaining probabilities (a copy of Mr. Butler's argument may be obtained from the editor). The final result is that the probability of an intersection is 0.896. When a computer simulation was run for 10,000 trials the result was 0.895.

Responses also received from Frank Rubin, Thomas Schonhoff, Anne Symanovick, David Emmes, Mary Lindenberg, and J. Jones.

J/J 4 The six digits in an automobile odometer occasionally show mirror symmetry — i.e., they are in the form xyz-zyx. How often does this happen during the complete 100,000-mile sequence through the odometer? Is there a particular 1,000-mile trip during which it would occur more often than on any other?

Frank Rubin sent us the following solution which handles both kinds of odometers — those with and without tenths of miles: For every three-digit combination xyz there is a unique reversal zyx. So there will be exactly 1,000 such palindromes. The distance between consecutive palindromes is determined by the number of digits that change. If $z < 9$, then two digits will change and the distance will be 1,100 miles; if $z = 9$ and $y < 9$, four digits will change and the distance will be 110 miles; if $y = z = 9$ and $x < 9$, six digits will change and the distance will be 11 miles; and if $x = y = z = 9$, then six digits will change and the distance will be one mile. Thus there are many trips that will include exactly two palindromes — namely, any trip whose start is less than a number of the form xy99yx by no more than 1000 − 110 = 890 miles. No trip can have three palindromes, since the minimum distance is 1,101 miles, from 999999 to 001100. I am here assuming the odometer registers miles, as in a Volvo, rather than tenths. Otherwise 11 palindromes can occur, say starting somewhere before 19999.1 miles and ending somewhere after 20990.2 miles.

More details on the non-Volvo case come from John Prussing: The maximum number of mirror sequences occuring during a 1,000-mile trip is 11. As an example, the trip between odometer readings of 09998.0 and 10998.0 contains the 11 palindromes: 09999.0, 10000.1, 10110.1, 10220.1, 10330.1, 10440.1, 10550.1, 10660.1, 10770.1, 10880.1, and 10990.1.

Starting from 00000.0, the palindromes occur every 110 miles. This pattern is restarted every 1001 miles and every 10000.1 miles. The palindrome at 1001 miles is 11 miles after the previous palindrome 00990.0 miles. The palindrome at 10000.1 miles is only 1.1 miles after the previous one. The 1000-mile trip containing the most palindromes must contain one of the 1.1 mile increments.

Also solved by Bruce Drew, Winslow Hartford, Jerome Taylor, Frank Carbin, and Avi Ornstein.

J/J 5 What is the longest English word (no proper nouns or chemical compounds, please) in which no letter occurs more than once — that is, all the letters in the word are distinct?

As usual with these problems, the real question is how outlandish should be the permitted winner. Surely everyone will grant the 14-letter AMBIDEXTROUSLY, found by Harry Hazard and the proposer, Kenneth Wise. Mr. Hazard, who must browse through Webster's *New International* for fun, found the 15-letter DERMATOGLYPHICS listed there. Curiously, he notes that UNCOPYRIGHTABLE (15) is *not* listed whereas Frank Rubin submitted this word and its 16-letter plural. Finally Alan Stiehl claims that VODKATHUMBSCREWINGLY (20) describes the application of thumbscrews by a drunken torturer.

Entries in the sweepstakes also came from Winslow Hartford and Avi Ornstein.

Better Late Than Never

NS 14 Jerry Griggs has submitted the following:
The answer obtained by Eric Jamin is apparently correct — at least it agrees with the answer given in W. W. Rouse Ball's *Mathematical Recreations and Essays.* Ball gives a nice presentation for the case in which the dominoes have number 0, 1, . . . , n for n = 4. The ordinary domino problem is n = 6. For n = 4, the answer given is 126720. For n = 8, the answer given is $(2^{35})(3^{13})(5^3)(711)(40787)$. An older reference, given by Ball, is for an algebraic approach to the problem in M. Reiss, *Annali di Matematica,* Milan, 1871, vol. 5, pp. 63-120.

PERM 3 Harry (Hap) Hazard has obtained partial results on the John Rule challenge mentioned in the introduction. Mr. Hazard writes:

The possible four-digit combinations can be divided into three groups; those with repeated digits, those without repeated digits but using zeroes, and those with neither repeated digits nor zeroes, exemplified respectively by the dates 1979, 1980, and 1978. I have examined all 126 combinations of the third type (which I considered the most promising of the three), with the following tentative results.

(1) I found no combination meeting the problem requirements.

(2) The best combination was 1-3-8-9 (or 1983, as a date to look forward to), which lacks only one: 79.

(3) The next best was 2-3-4-7, which lacks two: 58 and 87.

(4) Two combinations lack three: 1-3-7-9 (52, 55, 71) and 2-3-7-9 (49-59-67).

(5) Seven combinations lack four: 1-4-7-9, 2-3-4-8, 2-3-4-9, 2-3-6-8, 2-3-8-9, 2-4-5-7, and 2-4-6-7.

(6) The combination 1-2-7-8 lacks five, but none of these is under 91, so this has the longest starting run I've found (1-90).

(7) Four combinations will make all number from 1 through 100 if it were permitted to use only two or three of the four digits. Thus 2-3-8-9 lacks 69 (9x8-3, discarding 2), 74, (9x8+2, discarding 3), 82 (discarding 3 and 9), and 100 (98+2, discarding 3). Similarly, 2-5-8-9 lacks six (28, 70, 77, 80, 90, and 100, all of which can be formed with two or three digits), 3-5-7-9 lacks nine (52, 68, 73, 76, 86, 90, 92, 93, 97), and 5-6-7-8 lacks eleven (23, 24, 38, 42, 47, 52, 62, 68, 75, 91, and 92).

(8) The worst combinations found were 1-7-8-9 (lacking 23), 2-4-6-8 (also lacking 23), and 4-5-8-9 (lacking twenty). 1978 was a real stinker!

Y1978 Claude Ducret has responded.

1979 FEB 2 Joseph Steranka has found an alternate method of solution.

FEB 4 There seem to be some problems with the published solutions. Stan Lukzic notes that, starting from 0, the sequence ARC COS TAN produces an error. David Kessel notes that using the HP-45 one can do better than the results shown for at least three numbers:

Two: Cos e^x x^2 ln
Ten: Cos (f) 10^x
Nine: start with 10 (f) GRD (f) →DMS

Finally, Frank Rubin writes, I thought there must be something wrong with Smith Turner's solution because there is no "E" key on my TI-SR50. (I had only been able to produce 1, 2, 4, and 8 when I had tried the problem myself.) I knew something was really wrong — not just a typo — when I saw that he had obtained 7 by squaring the sequence for 3. Even I know $3^2 \neq 7$. So I tried his sequence for producing 3, using every possible key in turn for the missing "E." Nothing worked. I got either an error or π^8. I have now found a valid sequence for producing 3:

Key	*Display*	*Key*	*Display*
e	1	y^x	e^{e^2}
e	e	RCL	e
STO	e	=	e^{e^3}
x^2	e^2	ln	e^3
e	e^{e^2}	ln	3

The same method can be used to develop 5, just pressing x^2 twice in succession in the sequence above. Of course 6 and 9 are 3! and 3^2. This leaves 7 and 10.

J/J SD1 Waltvant Buser feels that the arguments given don't work since the sets are infinite. I stand by those answers although they might benefit from a few limits to infinity. But there is no time for infinite arguments in a speed problem.

Proposer's Solutions to Speed Problems

Both solutions this month are courtesy of the editor:

SD1 $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ is true when you restrict a and b to be positive reals. In general the square root is multiple-valued and more care is needed.

SD2 The average of the combined region is *not* the average of the two individual averages. Instead, it is the weighted average of these two averages. When Jones moves he increases the weight of the lower average and decreases the weight of the higher.

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