Puzzle Corner
**Allan J. Gottlieb**

*Allan Gottlieb is associate professor of mathematics and coordinator for computer mathematics at York College of the City University of New York; he studied mathematics at M.I.T. and Brandeis. Send problems, solutions, and comments to him at the Department of Mathematics, York College, Jamaica, N.Y., 11451.*

## Smooth Rolling for a Square Wheel?

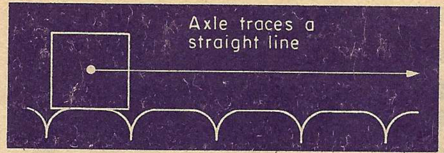
Let me begin by answering a perennial question. My method of selecting solutions to publish is as follows: As responses arrive during the month they are simply put together in neat piles with no regard as to their postmark or date of arrival. When it is time to write, I first weed out erroneous or illegible solutions. For difficult problems this may be enough. Usually, however, many responses still remain. I next try to select solutions that supply an appropriate amount of detail and that include a minimal number of characters that are hard to set in type. A particularly elegant solution is of course preferred. Of those selected, I favor contributions from respondents whose solutions have not previously appeared. Finally, especially neatly written or typed solutions are preferred, as this tends to reduce typographical errors.

### Problems

**OCT 1** We begin with a theoretical bridge problem by John Rutherford concerning the conditional probability of various card distributions. Mr. Rutherford writes what he calls "a serious question":

Suppose you and dummy have two seven-card fits (i.e., suits in which you are lacking six cards). The *a priori* odds for the division of six outstanding cards are well known at 48:36:15:1 for a 4-2:3-3:5-1:6-0 split. My question is, How do those odds change for the division of the second suit after you have played the first suit and established what the first split was? In particular, what are the odds for the division(s) of the second suit when the first suit is known to have split (a) 4-2 or (b) 3-3? I think they are quite different in these two cases, but before going on I would like to know what *you* think the probabilities are.

**OCT 2** Our second problem, from R. Crandall, sounds like one Conrail has been working on. Mr. Crandall wants you to describe a track upon which a square wheel rolls smoothly without slipping.




**OCT 3** Emmett Duffy likes to play with numbers. He would like you to find the smallest number N which can be partitioned into seven distinct positive integers such that the sums of any six of these integers is a perfect square. That is,

$$N = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7$$

and the following seven numbers are all perfect squares:

$$\begin{array}{l} \phantom{A_1 +{}} A_2 + A_3 + A_4 + A_5 + A_6 + A_6 \\ A_1 \phantom{{}+ A_2} + A_3 + A_4 + A_5 + A_6 + A_7 \\ \quad \vdots \\ A_1 + A_2 + A_3 + A_4 + A_5 + A_6. \end{array}$$

When you finish this try the sum of eight positive integers with the sum of any seven a square, then any eight out of nine, and finally any nine out of ten.

**OCT 4** John Rule sends us one of his favorites which he attributes to J. Pennington. It first appeared in the *American Mathematical Monthly* a quarter century ago:

SMITH: Down in Todd Country, which is a 19-mile square, I have a ranch — rectangular, not square, in shape — measuring a whole number of miles each way.

JAMES: Hold on a minute. I happen to know the area of your ranch; let me see if I can figure out its dimensions. (He figures furiously.) I need more information. Is the width more than half the length?

*Smith answered the question.*

JAMES: Now I know the dimensions of your ranch.

BROWN: I, too, know the area and, although I did not hear your answer to James' question, I, too, can tell you the dimensions.

GREEN: I did not know the area of your ranch but, since I have heard this conversation, I can deduce it.

What are the dimensions of the ranch?

**OCT 5** We close with a "best possible" cryptarithmetic problem from Frank Rubin: Replace each letter with a unique digit.

FOUND × A = SKILL

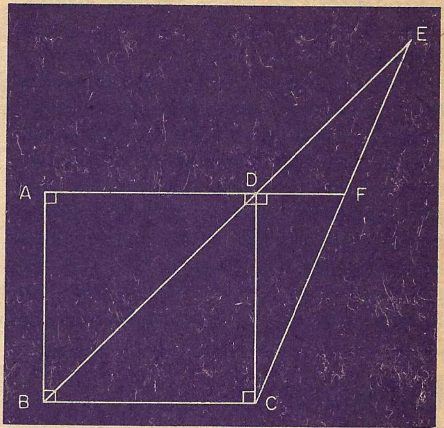
What is the value of SKILL? This is the *only* short multiplication in base 10 which has a unique solution and which involves as few as eleven digits in total. That is, this is the best possible problem, in the sense of least redundancy.

### Speed Dept.

**OCT SD1** Hugh Steward wants a common English word containing the sequence LIH.

**OCT SD2** Scott Byron has a cute geometry problem:

ABCD is a square. D is on BE. AB = BC = CD = AD = DE = 3. Find DF.



## Solutions

**MAY 1** White has four pieces — king, queen, and both bishops — all in starting positions on the first rank. Black has a lone king on Black's K4. White is to mate in three.

The following solution is from Jacob Dolid and his friend Boris:

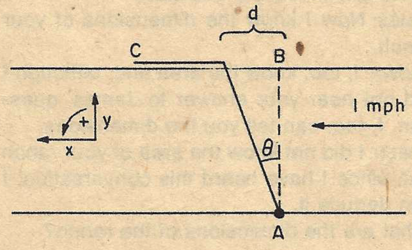
I thought you might be interested in learning how a solution was achieved. I am vacationing at my son-in-law's home and I have with me the electronic chess player, Boris. I introduced the positions specified and programmed Boris for a ten-minute interval and had him play the first move. He succeeded in making the key move of KB — B4. His move of Black's King to B3 permitted mate in three moves, as outlined below. This gave me the clue to the solutions for all possible moves.

| | *White* | *Black* |
|---|---|---|
| 1 | KB — B4 | K — B3 |
| 2 | Q — Q6 ch | K — B4 |
| 3 | Q — K6 mate; | |
| or | | |
| 2 | | K — N2 |
| 3 | Q — R6 mate. | |
| If | | |
| 1 | | K — B4 |
| 2 | Q — R5 ch | K — B3 |
| 3 | Q — N5 mate; | |
| or | | |
| 2 | | K — K5 |
| 3 | Q — Q5 mate. | |
| If | | |
| 1 | | K — K5 |
| 2 | Q — Q5 mate. | |

Also solved by George Colpitts, John Fine, J. Goldsmith, Glen Iba, Michael Jung, Ray Kinsley, Temple Patton, David Rabinowitz, Paul Reeves, Blaine Rhoades, Frank Rubin, Theodore Sauppe, Larry Shiller, Jerome Taylor, John Trifiletti, and the proposer, Elliot Roberts.

**MAY 2** A swimmer, who swims at a constant rate of two miles per hour relative to the water, wants to reach point C on the other side of the river from point A, where he now is. He swims from point A to point B, across the river, and then walks from point B to point C. You are to find the optimum angle ABC — i.e., the angle for which the travel time is minimized — for a given walking speed (which is greater than the swimming speed).

Steven Mayberry solved this completely and also calculated a percentage improvement for four-minute milers. His response follows:




The swimmer's speed in the y direction, in miles per hour, is (2 cos θ), and the time ($t_1$) to cross the river, in hours, is 1/(2 cos θ). The speed of the swimmer in the x direction is (2 sin θ); then adding the current of the river, his total speed (in miles per hour) is (2 sin θ + 1). Thus the distance that the swimmer is moved downstream (in the x direction) is

$$d = t_1(2 \sin \theta + 1).$$

The distance left to walk is (1 − d) and time to travel it is

$$t_2 = |(1 - d)/w|,$$

where w is the walking speed. The absolute value is necessary for the cases where d is greater than 1 mile between points B and C. That is where the swimmer would arrive downstream of point C and have to walk back. The total time spent ($t_T$) is $t_1 + t_2$. Substituting for $t_1$ and $t_2$,

$$t_T = 1/(2 \cos \theta) + |(1 - d)|/w,$$

and substituting for d,

$$t_T = 1/(2 \cos \theta) + |[1 - (2 \sin \theta + 1)/2 \cos \theta]|/w.$$

The next step is to take the partial derivative with respect to θ. To simplify the derivative, the absolute value is dropped for now:

$$\frac{\alpha t_T}{\alpha \theta} = \frac{\sin \theta}{2 \cos^2 \theta} - \frac{(2 \sin \theta + 1)\frac{\sin \theta}{2 \cos^2 \theta} + \frac{2 \cos \theta}{2 \cos \theta}}{w}$$

$$= \frac{\tan \theta}{2 \cos \theta} - \frac{\tan^2 \theta + \frac{\tan \theta}{2 \cos \theta} + 1}{w}$$

Setting this partial equal to 0 then yields the minimum except in the case where (1 − d) < 0. When this is the case, the partial does not reach a minimum; however, $t_2$ is negative, an impossible situation. If $t_2$ is always positive, any point that the swimmer arrives downstream from point C yields a longer travel time than no walking at all.

So, d must remain less than 1, else the solution for minimum time becomes the simple solution which specifies that points A and B are opposite each other on the two sides of the river. Thus, the time is minimum when the following two equations are satisfied:

$$\frac{\tan \theta}{2 \cos \theta} - \frac{\tan^2 \theta + \tan \theta/2 \cos \theta + 1}{w} = 0,$$

i.e., $\theta = \sin^{-1}(2/(w - 1)$,
when $(2 \sin \theta + 1)/2 \cos \theta \leq 1$ and
$\cos \theta - \sin \theta = ½$,
when $(2 \sin \theta + 1)/2 \cos \theta > 1$.

It is interesting to note that:

1. The solution to cos θ − sin θ = ½ is θ = ~24.295°, and $t_T$ is then 32 minutes 55 seconds.
2. For walking to make any difference in time at all (in contrast to swimming directly from A to C), w > 5.86 miles per hour; otherwise the θ to minimize $t_T$ all yield a distance downstream from point C and it is faster to swim direct; 5.86 miles per hour is pretty brisk to be called a walk — probably more in the class of jogging.
3. At w = 15 miles per hour (four-minute mile), θ = ~8.21°; $t_T$ = 31 minutes 43 seconds, a saving of only 3.7 per cent. Few people can run this fast. Many more can swim at 2 miles an hour.

Also solved by Gerald Blum, Irving Hopkins, Raphael Justewicz, Ray Kinsley, James Landau, Victor Newton, John Prussing, Paul Reeves, Frank Rubin, Larry Shiller, John Trifilleti, Don Uhl, Frederick Vose, and Harry Zaremba.

**MAY 3** Complete the following 3 × 3 magic square.

| | | |
|---|---|---|
| 1 | 2 | a |
| 3 | b | c |
| d | e | f |

Many readers were able to solve this one. The trick is not to assume that the entries are integers. The following complete derivation is from Cushnie:

| | |
|---|---|
| First row | (1) = 1 + 2 + a |
| Second row | (2) = 3 + b + c |
| Third row | (3) = d + e + f |
| First column | (4) = 1 + 3 + d |
| Second column | (5) = 2 + b + e |
| Third column | (6) = a + c + f |
| Diagonal l. to r. | (7) = 1 + b + f |
| Diagonal r. to l. | (8) = a + b + d |

(1) = (4): ~~1~~ + ~~2~~ + a = 1 + ~~3~~ + d
a = d + 1

(2) = (5): 3 + ~~b~~ + c = 2 + ~~b~~ + e
~~2~~ + 1 + c = ~~2~~ + e
e = c + 1

(7) = (8): 1 + ~~b~~ + f = a + ~~b~~ + d
~~1~~ + f = d + ~~1~~ + d
f = 2d

(3) = (7): d + e + ~~f~~ = 1 + b + ~~f~~
d + c + ~~1~~ = ~~1~~ + b
b = d + c

(1) = (2): ~~1~~ + ~~2~~ + a = ~~3~~ + b + c
~~d~~ + 1 = ~~d~~ + c + c
c = 0.5

(1) = (3): 1 + 2 + a = d + e + f
3 + ~~d~~ + ~~1~~ = ~~d~~ + c + ~~1~~ + 2d
3 = 0.5 + 2d
d = 1.25

a = d + 1 = 2.25
b = d + c = 1.75
e = c + 1 = 1.5
f = 2d = 2.5

| | | |
|---|---|---|
| 1 | 2 | 2.25 |
| 3 | 1.75 | 0.5 |
| 1.25 | 1.5 | 2.5 |

The answer is shown at the bottom of the previous column, with all eight totalling 5.25.

Also solved by Richard Bator, A. Bigus, Gerald Blum, Nancy Burstein, Frank Carbin, George Colpitts, Jack Crawford, Walter Delashmit, Doug Delgatty, Reverend George Doskocil, Robert Elkus, John Fine, Gertrude Fox, Clarence Gregory, Michael Haney, Winslow Hartford, Harry Hazard, Deborah Hooper, Glenn Iba, Michael Jung, Raphael Justewicz, Ray Kinsley, Joe Kaskal, Glen Krc, James Landau, Stephen Laug, Mary Lindenberg, Judith Longyear, Larry Marden, Naomi Markowitz, Sam McCluney, Roger Milkman, Avi Ornstein, Ronald Ort, Bert Posthill, Paul Reeves, Frank Rubin, Dan Sheingold, Jerome Shipman, Norman Spencer, John Trifiletti, William Turner, Charles Wolf, Harry Zaremba, and the proposer, Peter Groot.

**MAY 4** Find a continued fraction expansion for $\sqrt{x}$. That is a continued fraction written in terms of x that, for each positive x, converges to $\sqrt{x}$.

This problem is not easy. It does not suffice to give continued fractions for specific values of x. Of course using a term like $\sqrt{x+1}$ is forbidden. Judith Longyear (she now refers to me as Mr. Doctor Alice Gottlieb — presumably a take-off on the German "frau Doktor") submitted a derivation which she claims would be well known to "addicts of the Pell equation"; combining hers with the work of Paul Reeves and Glen Iba gives the following:

Let y be the desired continued fraction; then $y^2 = x$.

$$y^2 - 1 = x - 1$$

$$(y+1)(y-1) = x - 1$$

$$y - 1 = (x-1)/(y+1) \qquad \text{(A)}$$

$$y = 1 + \frac{1}{1/(x-1) + y/(x-1)}$$

$$y = 1 + \frac{1}{2/(x-1) + (y-1)/(x-1)}. \qquad \text{(B)}$$

But from (A) we have

$(y-1)/(x-1) = 1/(1+y)$. Thus

$$y = 1 + \frac{1}{2/(x-1) + 1/(1+y)}, \text{ or}$$

$$y + 1 = 2 + \frac{x-1}{2 + \frac{x-1}{y+1}}$$

As a continued fraction this becomes:

$$\sqrt{x} = 1 + \cfrac{x-1}{2 + \cfrac{x-1}{2 + \cfrac{x-1}{2 + \cfrac{x-1}{2 + \ldots}}}}$$

But now the question is, For what values of x does the limit converge? That is, define

$$a_0 = 2$$

$$a_1 = 2 + \frac{x-1}{2}$$

$$a_2 = 2 + \frac{x-1}{2 + \frac{x-1}{2}}$$

$$a_{n+1} = 2 + \frac{x-1}{a_n}$$

Is it the case that for each $x > 0 \lim_{n\to\infty} a_n$ exists? If it exists, the Reeves-Iba argument shows that the limit is $\sqrt{x} + 1$ as desired. For $x = 1$ everything is trivial. Assume $x > 1$. Then, as noted by Neil Cohen,

$a_1 < a_3 < a_5 < \ldots$ and $a_2 > a_4 > a_6 > \ldots$. Also $a_{2n-1} < a_{2n}$.

Thus the odd a's are growing and are less than the even a's which are shrinking. So each subsequence has a limit. All that is needed is to show that $(a_{2n} - a_{2n-1})$ approaches 0. This does not appear to be trivial. James Landau notes that this holds for $x < 2$. In fact he considers the parameterized continued fractions

$$b + \cfrac{x - b^2}{b + \cfrac{x - b^2}{b + \ldots}}$$

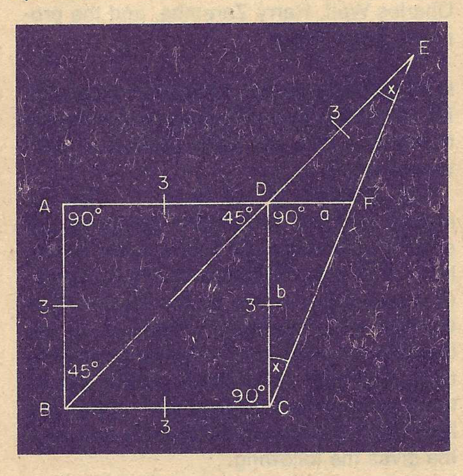
and concludes that this continued fraction approaches $\sqrt{x}$ for $b < x < 2b$. The Cohen analysis corresponds to $b = 1$. (A copy of Mr. Landau's analysis may be obtained from the editor. Mr. Landau also noted that other approximations to $\sqrt{x}$, in particular Newton's method, converge quicker.)

Responses were also received from Raphael Justewicz, Walter Penney, Frank Rubin, Norman Spencer, John Trifiletti, and the proposer, Ken Austin.

### Better Late Than Never

**1978 D/J 3** John Fine and Sidney Markowitz have responded.

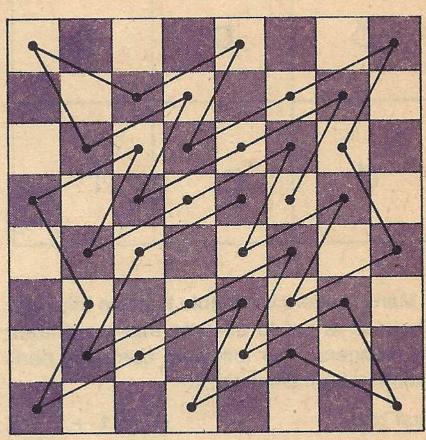
**1979 FEB 1** Frank Rubin has found the following *uncrossed* Knight's tour consisting of 35 moves (36 squares) which he claims is optimal:




### Proposers' Solutions to Speed Problems

**OCT SD1** LIKELIHOOD.

**OCT SD2** Most people don't see that angle DCF equals angle DEF and start constructing lines. This is unnecessary:



$135° + (2x)° = 180°$
$(2x)° = 45°$
$x = 22.5°$
$DF = a$
$b = DC = 3$
$\tan 22.5° = a/b = a/3$
$a = 3 \tan 22.5° = 1.242640687.$

My chairman, Joseph Malkevitch, happened to glance at this solution and immediately found a simpler method. He noted that $BD = 3\sqrt{2}$ and that, by similar triangles, $3/(3 + 3\sqrt{2}) = a/3$. Thus $a = 3(\sqrt{2} - 1)$.