How's Your Calculus?

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As a new volume begins, let me review the ground rules of "Puzzle Corner" for new readers.

Each issue we publish five regular problems and two "speed" problems. Three issues later the solutions to the regular problems appear. This month, for example, we are printing the solutions to problems published last May. Challenges to published solutions and acknowledgement of late responses appear in the "Better Late Than Never" department. The "speed" problems are not to be taken too seriously. Often whimsical, their solutions are usually given the same issue as the problem is posed, and they rarely appear in the "Better Late Than Never" department.

Here is some news from our readers:

I remember that during my senior year at M.I.T., many of the graduating seniors were considering their chances for acceptance at various graduate schools. One of my friends, Mike Rolle, decided to enhance his chances by solving the famous four-color conjecture. Since generations of mathematicians had failed in this attempt, we didn't feel that Mike had much hope of success; but he was serious. During that year he actually obtained some impressive partial results, but the conjecture was still unsettled. The end of this story occurred this year after Appel and Haken finally solved the problem. I was reading their important papers in the Illinois Journal of Mathematics and noticed an acknowledgement to one Michael Rolle for his help. Congratulations, Mike; he who solved the problem. I was reading their important papers in the Illinois Journal of Mathematics and noticed an acknowledgement to one Michael Rolle for his help. Congratulations, Mike; he who solved the problem. I was reading their important papers in the Illinois Journal of Mathematics and noticed an acknowledgement to one Michael Rolle for his help. Congratulations, Mike; he who solved the problem.

Problems

OCT 1 William Butler wonders how normal is normal (in bridge, at least): Conventional point counting gives four points for each ace, three per king, two per queen, and one per jack. The average bridge hand has ten such points. What is the probability of receiving a hand with exactly ten points?

OCT 2 Sebastian Batac would like to find two positive rational numbers the sum of whose cubes is 6. In other words, find positive integers a, b, c, and d satisfying (a/b)^3 + (c/d)^3 = 6.

OCT 3 The following cryptarithmic problem from Avi Ornstein consists of two mathematical statements which are correct in base 10 when digits are substituted for letters and is also true as read for modulo 9 mathematics:

SIX + SIX = TWO + ONE
SIX + SIX = TWO + ONE

OCT 4 J. Friedman sends me a number of problems published by Calibrom Products as advertising in Technology Review; this one appeared in 1938:

\[ \text{NS 12 A standard deck of 52 cards is shuffled and placed face down upon the table. The cards are then turned face up one at a time by flipping over the top card of the face-down stack. As this is done, the player simultaneously calls out the sequence A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A, 2, etc., one call being made for each card flipped over. To win the game, one must go through the deck without matching a card flipped over with the card called. Suits don't matter, so, for example, any 4-spot flipped over on the 4th, 17th, 30th, or 43rd turn results in a loss. "Since winter will surely come again," Mr. Conbine would like to know what are the chances of winning the game. How about a second solution for the same game with a 48-card pinochle deck? This problem is not trivial! Judith Longyear gave a colloquium talk on her results in 1974. The answer is not (12/13)^52. Although the probability of success for any one card is 12/13, the events are}

Starting with any triangle, construct three exterior triangles having base angles of 30° and vertices at D, E, and X — as indicated in the diagram. If the distance DE is taken as 100, what is the distance DX? (The answer is a definite number, not a formula.)

OCT 5 How's your calculus? Harvey Elentuck asks for the area of the loop of \( Y^2 = (X + 4)(X^2 - X + 2Y - 4) \). A noncalculus solution to this would be very impressive, but calculus is permitted.

Speed Department

OCT SD 1 Ruth Duffy asks us to name a word in the English language with seven letters, five of which are the vowels a, e, i, o, and u (but not necessarily in alphabetical order).

OCT SD 2 The Editor of the Review discovered the following problem being distributed as part of a tongue-in-cheek "quiz" prepared by M.I.T. students for exhibitors in the 1978 Massachusetts Science Fair. Translate into a limerick:

\[ (12 + 144 + 20 + 3\sqrt{2})/7 + 5 \times 11 = 9^2 + 0. \]

Translations

NS 10 (This was first published as 1974 M/A 2 and never solved; it was published again as NS 10 in February, 1978): Find a closed form for \( 1^2 + 2^2 + \ldots + n^2 \).

When this was first published Leo Epstein supplied some asymptotic formulas. He has improved these, but we still have no exact closed form. Perhaps none exists.

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not independent. Bob Kimble and an un-named computer assert that of the $52!$ possible decks exactly 1, 309, 302, 175, 551, 177, 162, 931, 045, 000, 000, 000, 000, 000. They obtain a success probability of about 1.62 per cent.

They also solved the pinochle problem: of the 48! decks, 2, 173, 013, 719, 746, 911, 580, 113, 686, 677, 997, 894, 282, 336, 936, 761, 753, 600, 000, 000 are winners. Since $52! = 80, 658, 175, 170, 943, 878, 571, 660, 636, 856, 403, 766, 975, 289, 505, 440, 883, 277, 824, 000, 000, 000, 000, they obtain a success probability of about 1.62 per cent.

Stephen Flaum and a TI 58 used an iterative technique:

At each iteration the probability of losing on that iteration is calculated. In addition, the expected number of cards remaining, conditional on the assumption that the game is not lost on that iteration, is calculated. These expected values are used in subsequent calculations of the probability of failure.

This method requires fractions to be kept throughout. Flaum and TI actually divide our the fraction and use the approximating decimal. Perhaps this explains their answer of 1.77 and .0225 percent for pinochle.

A. Walther claims the answer is:

$$\left(\frac{1}{(\sum_{P=0}^{N}(-1)^P) \cdot 1/P!}\right)$$

or approximately $e^{-4}$ (i.e., over 1.8 per cent). His remarks follow:

Make on the table an array 13 blocks long and four blocks wide. Label the four rows with the names of the four suits. Turn the cards over, one at a time, and place them in the array, going from left to right and placing each card in the row matching its suit. After we have gone through the entire deck, we have on the table four rows of 13 cards, one for each suit in the deck. To win the game no card must be in its proper place — i.e., each row of 13 cards must be a “complete permutation.” A complete permutation is defined as a permutation in which none of the elements is in its proper place. The theory of complete permutations is developed on some sheets saved for me by E. L. O’Neill which you may want to share with interested readers. The ratio of the number of complete permutations to the total number of permutations for $m$ elements is

$$\sum_{P=0}^{m}(-1)^P \cdot 1/P!$$

This is very nearly $e^{-1}$; therefore, the answer to the problem is $e^{-4}$ — i.e., about one in 53. (A copy of the notes on complete permutations may be obtained from the editor on request.)

MAY 1 White to play and win:

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1. P - B7
2. K - Q5
3. K - Q7
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Several readers slipped up on this one. By playing 2K — B5 they allow a neat draw:

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1. K - N5
2. K - N7
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Joseph Seo, however, avoids this and finds a solution with only one line:

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1. P - B7
2. K - N5
3. P - B8 (Q)
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MAY 3 Given an $n$-by-$n$ checkerboard and $n^2$ checkers of $n$ different colors, and given that there are $n$ checkers of each color, is it possible to arrange all the $n^2$ checkers on the board such that no two checkers of the same color lie in the same row, column, or diagonal? (By “diagonal” is meant all the diagonals, not just the two main diagonals.) It turns out that for certain values of $n$ it is possible to so arrange the checkers; in this case we say a solution exists — e.g., $n = 1$. But for certain other values of $n$ such an arrangement is impossible — i.e., no solution exists. For which values of $n$ does a solution exist?

For some unknown reason, I published this problem twice: once as FEB 3 and now again as MAY 3. More surprising than this is the fact that no reader noticed my error; as soon as I saw the May issue I made ready for the slings and arrows. The responses to MAY 3 are consistent with the published solution to FEB 3 (see June/July, page 27). In short, an algorithm exists for $N = 6K \pm 1$ (i.e., $N$ not divisible by 2 or 3), and several readers assert (without proof) that no solution exists for the remaining cases. I repeat my comment of June/July: this looks like an NS problem for the 1980s.


MAY 4 Assign numerical values to each letter:

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1. P — B8 (Q) — Resigns
2. ... — R — N8
3. 3P — B8 (Q) — R — B8 ch
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“oanthaldehyde” are excluded, since they are simply variant spellings or variant pronunciations of the same word.

Dennis Kluk submitted a list which will be hard to beat. (I must add that some of his words are not in my vocabulary; perhaps I should have specified a small dictionary in which all words were required to appear.) Mr. Kluk’s list, which follows, comes from Word Ways, subtitled the journal of recreational linguistics: Aquintocubitalism, Blithesomeness, Chemotherapeutics, Demulscifaction, Emotionlessness, Frightfulness, Gastrrophotographics, Hedriophthalmus, Identification(s), Japaconitine, Kineasthetic, Limitableness, Methylhydrocapreine, Neopaleozoic, Oesophagostenosis, Premosrepresentation(s), Quinta(s), Revolutionally, Selectiveness, Treasonableness, Utopographer(s), Vindictiveness, Whenceforward, Xanthosiderite, Yourselves, and Zoosporeferous.

Also solved by Harry Hazard, Jacob Bermann, Emmet Duffy, Paul Hertz, and the proposer, Donald Forman.
This is a base-12 cryptarithmetic problem, and those solving it were reminded that duodecimal notation has two extra digits following 9 before reading the radix. For uniformity, these were specified to be “dek” (symbol X, numerical value equals decimal 10) and “el” (symbol e, numerical value equals decimal 11). Then the radix is “dozen,” or “do” for short.

Cryptarithmetic problems tend to be popular, and this one, with its base-12 twist, was no exception. Several readers asked for more such problems; the best way to achieve this is to send more in. The following solution is from Shirley Wilson:

12e
33J9e)4135391
3359e
59749
66e7x
327571
302411
254x0

1. H = 1
2. C = 0
4. Z = x
5. O = 9
6. Since A - R = 9 or 10 + A - R = 9, and since A ≠ x and A ≠ e, 10 + A - R = 9 and hence 10 + H - 1 - A = 9. Therefore, A = 3 and R = 5.
7. P = 4, E = A - 1 = 2, D = 8, Y = 7, and B = 6.

Thus, as many readers noticed, the substitutions are:

CHEAPRBYDOZN


As a check, let the dog start at intersection 3. Then

\[ P_3 = P_4 + P_4 + 1/4. \] (4)

The relations (3) are consistent with (4).


1977 DEC 5 Irving Hopkins is not happy with Raymond Kinsey’s solution. Mr. Hopkins’ comments follow the drawing (see above) which shows his solution:

From my solution, given (θ + ϕ) = 60°, K = M/a = cos 60° + (cos 2 60° + 1) \[ = 1.618034 \]

Given a = 8, M = 12.9442
\[ h = a(sin 60°) = 6.928 \]
\[ θ = arctan (6.928/16.9442) = 33.76° \]
\[ δ = 60° - θ = 33.76° \]
\[ β = 120° \]
\[ γ = 180° - 120° - θ = 60° \]
\[ α = arctan (h(M - 4)) = 37.76° \]
\[ β = 180° - 60° - γ = 120° \]

From this, θ + ϕ = 60°
\[ δ = α = β = 120°, \] not (γ) = (α + β) and (δ) = (θ + ϕ)
\[ (60) = (120) and (120) = (60) \]

Also, from the above, β = 82.24, and θ = 22.24, whence β cannot equal θ, as Mr. Kinsey claims. Hence the rest of the argument falls apart. Also, if we consider the simultaneous equations R/S = M/L and 2R/L = M/S, we can eliminate R and M, with the result that L^2 = 25^2, which is obviously not generally true. Or similarly, by eliminating L and S, we find M^2 = 2R^2, equally untenable.

Proposer’s Solutions to Speed Problems

SD 1 Sequoia.

SD 2 (courtesy of AG): Twelve plus one forty four; plus twenty plus three roots of four; divided by seven; plus five times eleven; gives nine squared and not a bit more.

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