All 26 Letters in One Great Word?

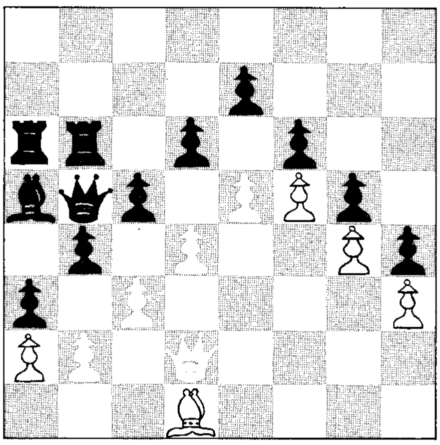
Puzzle Corner
by
Allan J. Gottlieb

Hello again. This month we are instituting a new policy suggested by Frank Rubin. Every other issue I will rummage through back issues and present a problem which has already appeared but was never (completely) solved. Perhaps the challenge of an (allegedly) difficult problem will be an inspiration. These problems will be labeled NS 1, NS 2, ... We begin with NS 1, which originally appeared in 1967.

Solutions to this month's problems will be published in the March/April issue; your responses should reach me by January 5.

NS 1 Let A be a non-empty set of Reals. Call $dA = \{a - b: a \in A\ b \in A\}$ the Distance set of A. What can be said about distance sets (measure, closed, connected, etc.)? What are necessary and sufficient conditions for a set, B, to be the distance set of a set?

DEC 1 We begin this month's regular selection with a chess problem from Mike G. Middlebrooke:



White to play and draw.

DEC 2 Frank Rubin wants to know the minimum total area of two circles which cover a unit square. How about three circles? Four circles?

DEC 3 Don Forman is interested in English words which have strings of consecutive letters. For example, PULMONARY contains LMNOP. What is the fewest number of words needed so that the consecutive strings use all 26 letters? (Only one string of consecutive letters is to be taken from each word.)

DEC 4 The following was submitted by Arthur Schach of the National Bureau of Standards: Take two unit squares and place them side by side so they form a rectangle two units wide and one unit high. Now choose a point A at random in the first square and, again at random, a point B in the second square. Then A and B will be a certain distance apart. Question: If we repeat the random choice of A and B many times, how far apart will they be on the average?

DEC 5 The following problem has an interesting corollary as pointed out by the proposer Eugene W. Sard: Show that there is an infinity of trigonometric functions of real numbers that are algebraic numbers. Specifically, given any real number a, where cos a is an algebraic number; if T is any direct trigonometric function, show that T(ma/n) is an algebraic number for all integers m and n. Mr. Sard writes that the implications of this problem "amaze" him: For example, since all the angles in a normal table of natural trigonometric functions are rational numbers of say 30°, all the corresponding function values are algebraic rather than transcendental numbers as he had always assumed. Even Tobias Dantzig in his book *Number, The Language of Science* says that most logarithms and trigonometric ratios are transcendentals.

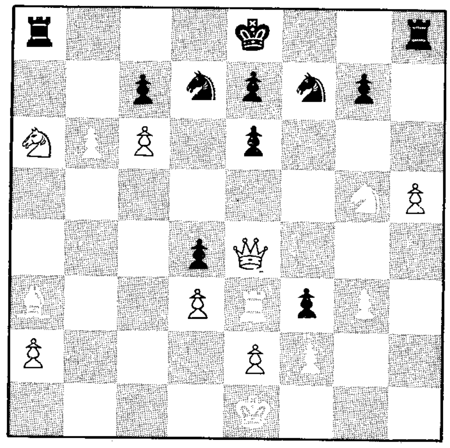
Speed Department

SD 1 R. Robinson Rowe remembers that the Matherville, Mathechusetts, Math Club meets periodically and more often than once a year — always on the 31st day of the month. When is its next meeting?

SD 2 Mr. and Mrs. Victor W. Sauer want you to "add five odd numbers to obtain 14."

Solutions

JUN 1 White to move and mate in two. *(See chessboard at top of next column.)*



It seems to me that Richard I. Hess and William J. Butler, Jr., are correct in their two conclusions:

(1) No mate in two is possible. If Q x P (K6), Castle KR ends the threat; N x P (K6), R x N also seems to lead nowhere. If only White's bishop was on R4, N x P (K6) R x N P x N would be mate. But, alas, it's only on R3.

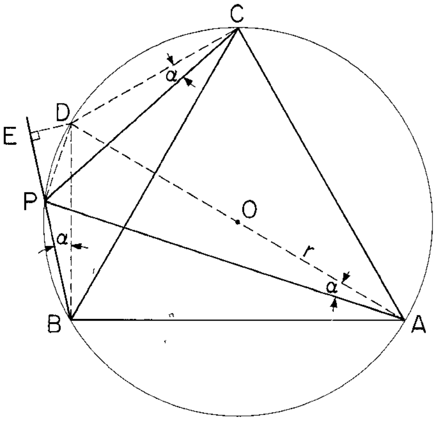
(2) The position cannot be reached in a game. Black has made three pawn captures but White has lost only two pieces.

Responses were also received from Sterling Watson, Herbert Moeller, Robert Aronoff, Jack Hiatt, John Musgrove, Chip Lawrence, M. Michael Laufer, Ron Moore, Frank Danforth, Avi Ornstein, John M. Rand, A. LeBlanc, Gerald Blum, Paul Mailman, Joel Tepper, Eric Jamin, and Jerry (Troll) Schwartz.

JUN 2 ABC is an equilateral triangle inscribed in a circle, and P is a point chosen on arc BC. Prove that AP = BP + PC.

Robert Pogoff points out that this problem already appeared in "Puzzle Corner." Nonetheless, many fine solutions were received. The one presented is from Jack C. Page, who writes:

I find your geometry puzzle most interesting, primarily because I did not suspect that the relationship existed. Once identified it is easy to prove if one remembers the theorem, "Any angle inscribed in a circle is equal to one-half the intercepted arc," and its corollary, "Any angle inscribed in a semi-circle is a right angle." Construct a diameter of the circle through point A, the center O to point D. Drop a perpendicular from D to BP extended at E. By our theorem, angle DAP = angle DBP = angle DCP = ½ arc DP which we will define as α. By our corollary, angle DPA is a right angle. Thus AP = AD cos α = 2 r cos α.



As can be easily proven, BD and DC, being the chords of a $\pi/3$ arc, are equal to r, the radius of the circle AO. By our theorem, angle DPC = $\pi/6$. Since by our theorem angle DPB = $5\pi/6$, by subtraction angle DPE = $\pi/6$.

$$PC = DC \cos \alpha + PD \cos \pi/6$$
$$PB = DB \cos \alpha - PD \cos \pi/6$$
$$\overline{PC + PB = (DC + DB) \cos \alpha} = 2 r \cos \alpha = AP \text{ Q.E.D.}$$

Also solved by Richard Brady, Chip Lawrence, Ron Moore, Paul Mailman, Jack Parsons, William B. Fisher, Gerald Blum, Richard I. Hess, William J. Butler, Jr., Eric Jamin, Emmet J. Duffy, Jeff Kenton, Thomas O. Ryland, N. B. Maclaren, Mary Lindenberg, John T. and/or Janet M. Rule, Raymond Gaillard, Victor W. Sauer, Harry Zaremba, John E. Prussing, Ken Kiesel, Roger A. Whitman, R. Robinson Rowe, Robert Gottlieb, Dr. and/or Mrs. Zoltan Cs. Mester, Harvey Elentuck, Naomi Markovitz, Joseph G. Haubrich, Dave Taenzer, and the proposer, George T. Marcou.

JUN 3 Given: one local gas company's old-fashioned storage tank. It floats like a rigid balloon, open end down, on a water sump. Vertical guides restrain it sideways but let it move up and down as gas is pumped in or out. Problem in ten parts:
1. Serve up a proof without numbers that as the tank goes up the gas pressure inside goes down (or up, or remains constant).
2. Is the sump a cylindrical hole or an annular moat (and no fair asking anybody around the gas works; they have lost the blueprint)?
3. A. The tank is half full (or half empty), no gas is added or removed, but the barometer drops. Does the tank go up or down? (*Yes* is not an acceptable answer.)
B. What about the water level(s)?
4. Would it make any difference in *3* whether the hole was annular or cylindrical, or other?
5. Would it make any difference in *3, 9, 10,* if the hole were filled with mercury or olive oil instead of water?
6. Would the tank top be a good place for a penthouse? Or a heliport?
7. Could you employ an escalator to get there and back?
8. With a decorated tank that rotates for advertising purposes, what precautions are required against freezing?
9. If the tank never goes all the way to the top, was it built too large? Does evaporation let the gas company make money on vapor?
10. Same, if it never goes all the way to the bottom? How can the interior be checked out for corrosion?

R. Robinson Rowe claims first-hand experience:
That old-fashioned gas storage tank was called a "gas holder."
I know how it works because 70 years ago a miniature was set up in our basement on the farm, using acetylene generated from calcium carbide instead of "manufactured gas" from coking coal which filled the big one in town. The lower part was called the tank and was nearly full of water. The upper part was the hood and I will call its upper flat surface the deck and the cylindrical part its skirt. Since both kinds of gas are lighter than air and weight of gas enters the problem, it will be convenient to use absolute pressures per unit area in horizontal sections. And since numbers are banned, let

a = atmospheric pressure
w = unit weight of the deck in the same unit
s = equivalent unit weight of the unbuoyed portion of the skirt
u = uplift pressure of the gas in contact with the deck
d = downward pressure of the gas in contact with the water
h = height of the gas column between water and deck
g = weight of gas in a unit column = kh(u + d) where k is a constant
x = weight of water in a unit column as high as the inside-outside difference in levels.

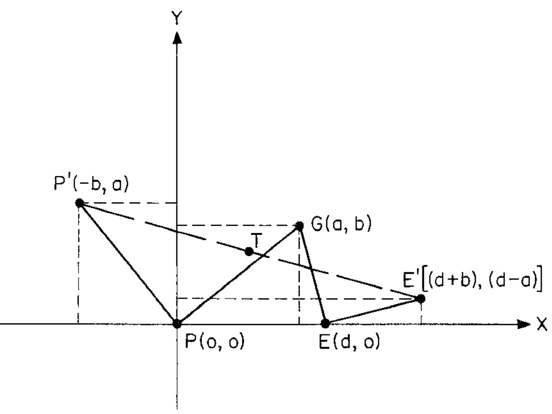
1. We have u = a + w + s. As the tank goes up, a and w are unchanged, but the skirt rises in the water, reducing buoyancy and increasing s. Therefore u, the gas pressure at the top, goes UP.
2. The sump is a cylindrical hole. The ones I've seen are only partly underground — probably just far enough to reach a solid foundation for the invert of the tank.
3. The downward pressure of the gas in contact with the water is the upward pressure on the deck plus the weight of the gas column between, or d ≐ u + kh(u + d) = a + w + s + kh(u + d). The downward pressure on the water outside the skirt is a. So the inside-outside difference in water levels x = d − a = w + s + kh(u + d). So when the barometer drops, reducing a, x remains unchanged. However, for a gas at constant temperature and mass, PV = C, that is, pressure times volume is constant. Now u and d each include a and diminish, so in the term kh(u + d) which remains constant, h must increase. This makes the hood tank go up. So subpart A is *UP* and subpart B is *unchanged*.
4. No.
5. No.
6. Poets described a city's slums as, "the other side of the railroad tracks and down by the vinegar works," but if the city had a gas holder, that's where the real slum was. Input gas filtered thru the water as a scrubber, polluting the water with sulfide and the less volatile hydrocarbons. An intolerable environment for a penthouse. Also the fire hazard makes it unsuitable for either penthouse or heliport; and for the latter, lightning rods around the hood would be hazards on a foggy night.
7. It might be possible to design an "elastic" escalator — but prohibitively expensive for the low traffic volume.
8. I never heard of the water freezing in a gas holder. The large volume of enclosed water must remain at a fairly constant temperature. The seal water between skirt and outer tank might freeze thin at the surface but not solid enough to retard a rotating skirt.
9. This is a question of both economic and prudent design. A utility facility should have a reasonably comfortable margin for peak demands and future growth. If the holder reached its top limit, I think the manager would wish it was 10 feet bigger. I doubt that there is a considerable hazard of overdesign; when you see a gas holder you probably see a plantation of them. As for the gas company profiting on evaporation loss, No, not any more than it profits on other inefficiencies inherent in the system. The customer is billed for the volume passing thru his meter at a rate set by a public utilities commission adjusted for the B.t.u rating and periodically revised to allow the company a limited return on its capital investment (known as its rate base).
10. Practically the same answer here, but the margin is more important. If in part 9 the holder reached the top, the inlet valve could be closed; but if it reached the bottom, irate customers would be eating cold cuts and inconvenienced by pilot lights going out. As for a corrosion check, cathodic protection could reverse galvanic action so that there would be a harmless encrustation instead of corrosion.
11. Why does Quilter want to know all these things? Is he writing a book? Or going into the gas business?

Also solved by Jack Parsons, William J. Butler, Jr., Joseph G. Haubrich, Gerald Blum, Richard I. Hess, and Thomas F. Snyder.

JUN 4 An M.I.T. student bought a treasure map from an old sea captain who told him the coordinates of the small island on which the treasure was buried. The map showed a palm tree, a eucalyptus tree, and an old wooden gallows. The instructions said to walk from the gallows to the palm tree, counting the number of steps. At the palm, turn right by a right angle and take the same number of steps, placing a stake in the ground at the point reached. Start again at the gallows and walk to the eucalyptus, counting the number of steps. At the eucalyptus turn left by a right angle and take the same number of steps, placing a stake at the point reached. The treasure is to be found buried half way between the two stakes. The sea captain told the student that the old gallows had completely disappeared, having rotted away; but the trees still stood. The student is attempting to devise a method for locating the treasure. Can you help him?

The key, of course, is that the location of the gallows is immaterial. Carl M. King submitted the following:

The information given will find the treasure, beginning at any arbitrary location of the gallows. The sketch shows the construction. The only measurement of significance is the distance (d) between the palm tree and the eucalyptus tree.



	Coordinates x	Coordinates y
P = palm tree	0	0
E = eucalyptus tree	d	0
G = gallows	a	b
P′ = first stake	−b	a
E′ = second stake	(d + b)	(d − a)
T = treasure	½ d	½ d

Using arbitrary coordinates for the gallows, G(a, b), you carry out the construction. When you calculate the coordinates of the mid-point between points P′ and E′, you obtain a result that is independent of the coordinates of point G.

The coordinates of T (treasure) are [(d + b− b)/2, (d − a + a)/2], for which x = d/2 and y = d/2. Isn't that surprising?

Also solved by Harry Zaremba, Claude Rabache, Robert Pogoff, Russell A. Nahigian, Raymond Gaillard, James N. Cawse, Richard S. Galik, Jack F. Parson, John T. and/or Janet M. Rule, Christian P. Marchand, Glen Ferri, T. H. Greenway, Neil E. Hopkins, Judith Q. Longyear, William Benton Fisher, Daniel E. Feldman, Jim Toker, Joel Tepper, Chip Lawrence, Paul Mailman, E. Jamin, Ken Kiesel, Jeff Kenton, W. A. Schoenfeld, Richard I. Hess, Gerald Blum, William J. Butler, Jr., R. Robinson Rowe, and the proposer, John E. Prussing.

JUN 5 Given any collection of straight streets $S_1, S_2, \ldots, S_K$ intersecting at points

$I_1, I_2, \ldots, I_n$, describe a general method of finding the placement of a minimum number of policemen so that every intersection can be seen by at least one policeman.

The following solution, consisting of three methods of increasing sophistication, was submitted by Richard I. Hess:

1. *Explicit enumeration:* Try all ways of placing m cops at the n intersections starting with m = 1, then m = 2, m = 3, etc., until a lowest m achieves a condition where each intersection is visible.

2. *Implicit enumeration or branch-and-bound:* Put a cop at each of the intersections; obviously they can see all intersections. Now consider removing a cop from each intersection in turn and remove from consideration all configurations which don't allow visibility of all intersections. Repeat this process of creating more configurations by removing single cops from each configuration in all possible ways. Eliminate those which are not solutions until none remains. The last configuration remaining is a minimal solution.

3. *Linear programming (recommended by friend, Joe Bartlett):* Define:

$T_{ij} = 1$ if placing a cop at node i makes node j visible;
$= 0$ if not.
$A_i = 0$ if node i has a cop there;
$= 1$ if not

Objective: minimize

$$X = \sum_{i=1}^{n} A_i .$$

Subject to

$$C_i = \sum_{j=1}^{n} T_{ij} A_j > 0 \text{ for all } i = 1, 2, \ldots, n$$

$$A_i \geq 0 \text{ for all } i = 1, 2, \ldots, n$$

This problem can be solved by standard linear programming methods.

Method 3 was also proposed by Hal Vose, and other responses were received from R. Robinson Rowe, William J. Butler, Jr., Dr. and/or Mrs. Zoltan Cs. Mester, and Joseph G. Haubrich.

Better Late Than Never

DEC 1 and DEC 2 Emmet J. Duffy has responded.

JAN 1 John M. Rand has responded.

M/A 2 Raymond Gaillard and E. Jamin have responded.

M/A 3 Eric Jamin, Lt. Col. Douglas H. Merkle, and John T. and/or Janet M. Rule have responded.

M/A 4 Gerald Blum, Winslow Hartford, Ken Kiesel, William R. Knowlton, Ralph Menkoff, Peter Ross, R. Robinson Rowe, and John Woolston.

M/A 5 Gerald Blum, Guy F. Boucher, Gregory C. Daley, Winslow Hartford, Ken Kiesel, R. Robinson Rowe, John Woolston, Raymond Gaillard and John T. and/or Janet M. Rule have responded. Emmet J. Duffy does not care for the published solution, which he says is not a gen-

eral one. He suggests:
To find three distinct positive integers such that the sum of any two is a square, take any three different squares whose sum is even, with the largest square less than the sum of the other two. Subtract the largest square from the sum of the other two and divide by 2 to obtain one integer. Subtract this integer from each of the two smaller squares to obtain the other two integers. For example, take 64, 144, and 196 as the three squares. Then $(64 + 144 - 196)/2 = 6$; $64 - 6 = 58$; $144 - 6 = 138$. The trio 6, 58, 138 cannot be found by any of the published methods. Let a, b, c be the three integers and M^2, N^2, O^2 be the squares in ascending order. Let $a + b = M^2$, $a + c = N^2$ and $b + c = O^2$. Then $2a + 2b + 2c = M^2 + N^2 + O^2$. Hence the sum of the squares must be given. Solving for a, we find that $a = (M^2 + N^2 - O^2)/2$. Hence for a to be positive O^2 must be less than $M^2 + N^2$. Then $b = M^2 - a$, and $c = N^2 - a$. The first solution shown in the July/August issue requires that one of the squares be $25k^2$ where k is an integer. For the second solution, if a is 2 and b is 7 their sum is a square and their difference is odd but the third integer is also 2. The third and fourth solutions require that the squares be the squares of numbers in arithmetical progression. Fault the second solution because it gives negative answers. If a is 10 and b is 15, then c is -6.

MAY SD 1 Jack Parsons disclaims responsibility, and Emmet J. Duffy and William J. Butler feel that the probability is greater than 7/16.

May 3 E. Jamin has responded.

Proposor's Solutions to Speed Problems

SD 1 There are seven 31-day months of the 12 in a year. The only aliquot parts of 12 are 2, 3, 4, and 6. No subset of the seven is spaced at two-, three-, or four-month intervals and only one subset {January, July} is spaced at a six-month interval. Therefore, they last met in July and will meet again next month.

SD 2

$$\begin{array}{r} 11 \\ 1 \\ 1 \\ \underline{1} \\ 14 \end{array}$$

[They saw this on TV so it must be o.k.-Ed.]

Allan J. Gottlieb, who studied mathematics at M.I.T. (S.B. 1967) and Brandeis (A.M., 1968, Ph.D. 1973), is Assistant Professor of Mathematics at York College of the City University of New York. Send problems, solutions, and comments to him at the Department of Mathematics, York College, 150-14 Jamaica Avenue, Jamaica, N.Y., 11432.