Start at the Gallows (Which Is Missing)

Puzzle Corner
by
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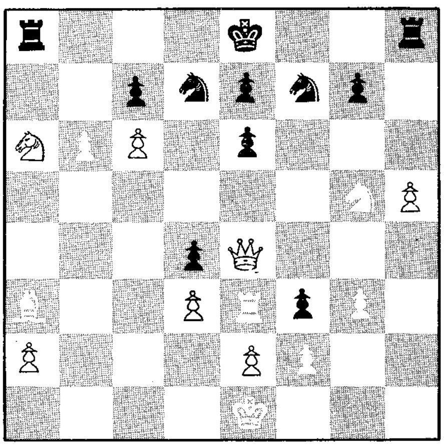
A few months ago I asked about reversing camera lenses for magnification ratios exceeding 1:1. Many readers responded. I am pleased to say that there is scientific justification for the practice. Briefly, lenses are optimized for the usual case where object size exceeds image size. When the opposite is true, reversing the lens causes the front of the lens to face the larger "thing" (the image) while the rear faces the smaller (the object), effectively returning us to the usual case. Many thanks to all who responded.

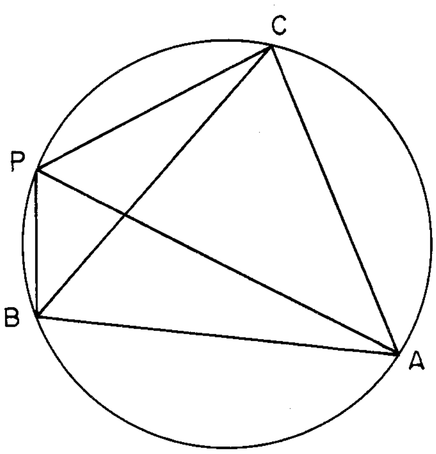
Problems

JUN 1 We begin this month with a chess problem from Harry Nelson:



White to move and mate in two.

JUN 2 A geometry problem from George Marcov:



ABC is an equilateral triangle inscribed in a circle, and P is a point chosen on arc BC. Prove that AP = BP + PC.

JUN 3 The proposer, Edward Quilter, submitted the following as a speed problem, but I feel it is more appropriate as a regular problem:

Given: one local gas company's old-fashioned storage tank. It floats like a rigid balloon, open end down, on a water sump. Vertical guides restrain it sideways but let it move up and down as gas is pumped in or out. Problem in ten parts:

1. Serve up a proof without numbers that as the tank goes up the gas pressure inside goes down (or up, or remains constant).
2. Is the sump a cylindrical hole or an annular moat (and no fair asking anybody around the gas works; they have lost the blueprint)?
3. A. The tank is half full (or half empty), no gas is added or removed, but the barometer drops. Does the tank go up or down? (*Yes* is not an acceptable answer.) *B.* What about the water level(s)?
4. Would it make any difference in *3* whether the hole was annular or cylindrical, or other?
5. Would it make any difference in *3, 9, 10,* if the hole was filled with mercury or olive oil instead of water?
6. Would the tank top be a good place for a penthouse? Or a heliport?
7. Could you employ an escalator to get there and back?
8. With a decorated tank that rotates for advertising purposes, what precautions are required against freezing?
9. If the tank never goes all the way to the top, was it built too large? Does evaporation let the gas company make money on vapor?
10. Same, if it never goes all the way to the bottom? How can the interior be checked out for corrosion?

JUN 4 John E. Prussing is looking for buried treasure: An M.I.T. student bought a treasure map from an old sea captain, who told him the coordinates of the small island on which the treasure was buried. The map showed a palm tree, a eucalyptus tree, and an old wooden gallows. The instructions said to walk from the gallows to the palm tree, counting the number of steps. At the palm, turn right by a right angle and take the same number of steps, placing a stake in the ground at the point reached. Start again at the gallows and walk to the eucalyptus, counting the number of steps. At the eucalyptus turn left by a right angle and take the same number of steps, placing a stake at the point reached. The treasure is to be found buried exactly half way between the two stakes. The sea captain told the student that the old gallows had completely disappeared, having rotted away; but the trees still stood. The student is attempting to devise a method for locating the treasure. Can you help him?

JUN 5 The following difficult problem from Frank Rubin was suggested by 1972 O/N SD2: Given any collection of straight streets $S_1, S_2, \ldots, S_k$ intersecting at points $I_1, I_2, \ldots, I_m$, describe a general method of finding the placement of a minimum number of policemen so that every intersection can be seen by at least one policeman.

Speed Department

SD 1 Our first speed problem is from W. D. Mohr: Three toggle switches may be manipulated, one at a time. A change in position of any of the three switches is called a move.

1. How many unique placements are there for the group of three switches?
2. How many moves are required to enter all possible placements?
3. From any given placement, how many moves can be made?
4. How many moves are required to make all possible moves?
5. Defining a "pair of moves" as two moves that can be made one right after the other, how many possible unique "pairs of moves" exist?
6. How many moves are required to make every possible "pair of moves"?

SD 2 We end with the following from Morry Markovitz: Take any two integers, either both odd or both even, whose sum is S, which is divisible by 3. Prove that the absolute value of the difference of the squares of these numbers is divisible by 2S/3.

Solutions

O/N 1 (as revised in February, 1975) Black and White are to cooperate to checkmate White *by discovery* in the fewest possible moves, starting from the standard beginning position. What are the moves if Black is constrained to move only one piece with which he may neither capture nor give check (he may, of course, mate with the piece)?

The shortest solution is the following, in ten moves, from Gerald Blum:

1	P-K3	N-KR3
2	B-Q3	N-N1
3	P-KN4	N-KR3
4	K-B1	N-N1
5	K-N2	N-KR3
6	K-N3	N-N1
7	BxP	N-KR3
8	B-K4	N-N1
9	B-N2	N-KR3
10	K-R3	N-B4

Also solved by Edward Ocampo, Paul Reeves, Stuart Schulman, and the proposer, Frank Rubin.

FEB 1 Can one deal a hand of bridge with which either side may make even two hearts?

Our best solution is from Daniel Pratt, who gives a hand where either side (but only one player from each side) can make three hearts against *any* defense:

North: ♠ — ♥ — ♦ 6 5 4 3 2 ♣ Q J 10 9 8 7 6 5

West: ♠ 9 8 7 6 5 ♥ 10 9 8 7 6 5 4 3 ♦ — ♣ —

East: ♠ 4 3 2 ♥ 2 ♦ A K Q J 10 9 ♣ A K 2

South: ♠ A K Q J 10 ♥ A K Q J ♦ 8 7 ♣ 4 3

With South as declarer, West leads either a heart or a spade, South wins in his hand, pulls East's trump (if still necessary), and takes all his spade and heart winners for his nine tricks. If West is declarer, North leads a diamond or club, West wins in Dummy with the ♦ A or ♣ A, leads the other ace, the two kings, and then the ♦ Q, discarding spades from his own hand on all five of these tricks. South can take only four tricks (his high trump), leaving the balance for West regardless of whether he trumps the ♦ Q lead.

Also solved by R. Robinson Rowe and Kenneth Lebensold.

FEB 2 Find the set of positive real numbers whose sum is 100 and whose product is maximal.

The following solution is from Allan Gottlieb (after reading everyone else's): Given n, it is fairly easy to show that the maximal value occurs when each number is 100/n (for n = 2 calculus, if $n > 2$ and $a_1 \neq a_2$, we can make the product larger by replacing a_1 and a_2 with $(a_1 + a_2)/2$. So the problem is to maximize $f(n) = (100/n)^n$ for *integral* n; if n is not integral, how can the set have n elements? We convert this to the problem of maximizing $(100/x)^x$ for real x. Again using calculus (differentiate and set the result equal to zero), one finds the solution to be $x = 100/e$. But we want the integral solution. Since x is between 36 and 37, most people tried each and chose 37 since it gives the larger result. To prove that this is the largest integral solution (which, in general, need not be near the largest real solution), one should point out that the above x is the *only* zero of f′ and that $f''(100/e)$ is negative. Hence f decreases for $x > 100/e$ and increases for $x < 100/e$; and therefore the maximal integral solution must be either 36 or 37.

Solutions received from Gerald Blum, Sy Comwill, Daniel Feldman, Lionel Goulet, Winslow Hartford, Jerry Hedden, Eric Jamin, Jeff Jordan, Kenneth Kiesel, Zalten Mester, Charles Musselman, Randall Neff, Daniel Pratt, John Prussing, Paul Reeves, R. Robinson Rowe, Frank Rubin, John Rule, Alfredo Sadun, W. A. Schoenfeld, Stuart Schulman, and Harry Zaremba. Randy Merilatt gave an interesting application to probability theory.

FEB 3 If the numbers from 1 to 5,000 are listed in equivalence classes according to the number of written characters (including blanks and hyphens) needed to write them out in full in correct English, there are exactly 40 such non-empty classes. There is a class with exactly one number; what is it?

The following solution is from Kenneth Kiesel:

None of the one-digit numbers is unique. Of the numbers 10 through 19 only 17 is unique; it is equivalent to 51. So 1 through 19 are eliminated. Any two-digit number above 19 whose last digit is nonzero cannot be unique since no digit is unique (x4 is equivalent to x5 and x9, etc.). The remaining two-digit numbers 20, 30, . . ., 90 are not unique by inspection. Thus there is no unique number from 1 through 99. No three-digit number is unique because no digit is unique (3xx is equivalent to 7xx and 8xx, etc.). No four-digit number that is not an even thousand is unique because none of the digits 1, 0, 9, 9, or 9 is unique. The only possibilities remaining are the even thousands. 1,000 is equivalent to 2,000 and 4,000 is equivalent to 5,000. The only possible unique number is 3,000, with 14 characters. But is it unique? The other even thousands have 12 or 13 characters. The shortest remaining four-digit number (e.g., 1,001) has 16 characters. Thus there are no four-digit numbers equivalent to 3,000. The longest even hundred (e.g., 300) has 13 characters. The shortest of the remaining three-digit numbers (e.g., 101) has 15 characters. Thus there are no three-digit equivalents. The longest two-digit number (e.g., 77) has 13 characters, and the longest digit has five. Therefore 3,000 is indeed unique.

Also solved by Steven Baum, Gerald Blum, Daniel Feldman, Winslow Hartford, Eric Jamin, Jeff Jordan, Randall Neff, Daniel Pratt, John Prussing, Paul Reeves, R. Robinson Rowe, Frank Rubin, Stuart Schulman, Harry Zaremba, and the Green Phantom (again!).

FEB 4 In the programming language BASIC, each line is numbered and the subroutine call is called GOSUB. It transfers control to a specified line number, as in 10 GOSUB 20.

Control continues as usual from there until a RETURN instruction is read, when control is passed back to the line *following* the GOSUB. When several GOSUBs are executed without intervening RETURNs, they are *stacked*; that is, a RETURN returns to the line following the *latest* pending GOSUB which is then removed from the stack. The next RETURN encountered refers to the previously pending GOSUB, which is then removed. And so on. Assuming a RETURN without pending GOSUB is illegal, can you prove the legality or illegality of this program:

```
10 GOSUB 20
20 GOSUB 30
30 GOSUB 40
40 GOSUB 50
50 GOSUB 60
60 GOSUB 70
70 GOSUB 80
80 GOSUB 90
90 RETURN
99 END
```

How many GOSUBs were executed?

A unanimous decision: the program is illegal! The following elegant proof is from Robert Mandl:

The program bombs out after $2^n - 1$ GOSUB executions, where n is the number of GOSUB statements in the program ($2^8 - 1$, or 255, in the case of the program given); it reaches the RETURN statement 2^n times, and the last time there is no GOSUB pending in the stack. The proof is by mathematical induction on the number of GOSUB statements. The induction starts at $n = 0$; the program is

```
90 RETURN
99 END
```

and it obviously bombs out the first time (thus the only time) the RETURN statement is reached. $1 = 2^0$. Suppose the program with k GOSUBs bombs out for lack of a pending GOSUB on the 2^kth time the RETURN statement is entered. After the first GOSUB statement of the program containing $k + 1$ GOSUB statements is executed, the sequence of events is identical to the execution sequence of the k-GOSUB program *except* for the fact that at all times there is an extra GOSUB return address at the bottom of the stack (corresponding to the extra GOSUB statement executed prior to entering the k-GOSUB portion). Thus the RETURN statement will be entered 2^k times. The original k-GOSUB program bombed out at this stage for lack of a pending GOSUB. The expanded program, however, still has one GOSUB in the stack; thus, rather than bombing out, it transfers control to the statement found in the line immediately below the line containing the stacked GOSUB — i.e., it is again at the beginning of the k-GOSUB program segment, but now with an *empty* stack of pending GOSUB requests. It will bomb out, therefore, just as the original k-GOSUB program did, after the RETURN statement is reached 2^k more times, for a total of 2^{k+1} times. Thus for all n (before the set of available statement numbers is exhausted), the n-GOSUB program bombs out after $2^n - 1$ instances of GOSUB transfer. (An additional solution could be based on the observation that the locus of control in the n-GOSUB program mimics closely enough the pattern of bit changes in an n-bit binary counter, where the 2^nth step results in an overflow.)

Also solved by Gerald Blum, Lionel Goulet, Jeff Kenton, Randall Neff, and Frank Rubin.

FEB 5 Can you build a 3 x 3 x 3 magic cube using the integers 1 through 27 once each? How about a magic hypercube using the integers 1 through 81 once each?

Again we have unanimous agreement: a magic cube is impossible, assuming that by magic cube we mean that each plane parallel to a face is a magic square. We proved several months ago that in a magic square the middle element must be one-third of the common sum. Since each plane will have the same sum, this one number would have to appear in many places. By weakening the hypotheses, several readers found solutions. The following is from Loren Dickerson:

The magic cube below is one of the entire set of possible 3 x 3 x 3 magic cubes described by W. S. Andrews (*Magic Squares and Cubes,* New York: Dover Books, 1960). All the rows, columns, "lines" (into the paper), and major diagonals have the magic sum, 42, except the two-dimensional diagonals of the outer squares. The cube also is "associated" in that all pairs of numbers diametrically equidistant from the central cell add up to 28, or twice the average of the series.

Top			*Middle*			*Bottom*		
1	17	24	23	3	16	18	22	2
15	19	8	7	14	21	20	9	13
26	6	10	12	25	5	4	11	27

The four-dimensional "hypercube" shown in the box at the bottom of this page was generated independently but is a variation of one attributed to Dr. C. Planck. The rules for its formation are as follows: The natural numbers 1 through 81 are divided into three series beginning with 1, 2, and 3. The series progress from these with intervals of four except immediately after multiples of 3, when the interval is 1. Placement from a series into cells of the squares is regular, moving to the corresponding cell in the square to the "northwest," except for three kinds of breakmoves corresponding to and occurring immediately after multiples of 3, 3^2, and 3^3, respectively, in the 1-series and after every third, ninth, and 27th numbers in the 2- and 3-series. The three series are shown with single, double, and triple virgules at the three respective kinds of breakmoves:

1, 5, 9 / 10, 14, 18 / 19, 23, 27 //
28, 32, 36 / 37, 41, 45 / 46, 50, 54 //
55, 59, 63 / 64, 68, 72 / 73, 77, 81 ///

2, 6, 7 / 11, 15, 16 / 20, 24, 25 //
29, 33, 34 / 38, 42, 43 / 47, 51, 52 //
56, 60, 61 / 65, 69, 70 / 74, 78, 79 ///

3, 4, 8 / 12, 13, 17 / 21, 22, 26 //
30, 31, 35 / 39, 40, 44 / 48, 49, 53 //
57, 58, 62, / 66, 67, 71 / 75, 76, 80 ///

The breakmoves are:

— After every third number: to the corresponding cell one square to the southwest, then one cell to the right of that square. After every ninth number: to the corresponding cell one square to the south.

— After every 27th number: one cell up (north) in the same square.

Note that the squares may be imagined to "wrap around" vertically, horizontally, and diagonally, so that moving a number three cells in any direction confined to the same 3 x 3 square returns the number to its original cell. The resulting hypercube has the magic sum of 123 in all directions parallel to the four dimensions, in its eight hyperdiagonals, and in several other diagonals. The sums of the numbers in the cells of each square is 369. This is true also for many 3 x 3 groups of adjacent cells in the 9 x 9 square, chosen randomly. The pairs of numbers diametrically equidistant from the central cell total twice the central number, or 82. The 9 x 9 square is associated in addition to being magic.

Also solved by Gerald Blum, Winslow Hartford, Roger Lustig, Paul Reeves, R. Robinson Rowe, Frank Rubin, and the proposer, Eric Jamin.

34	74	15	20	42	61	39	7	47
23	45	55	72	1	50	28	77	18
66	4	53	31	80	12	26	39	58
65	6	52	33	79	11	25	38	60
36	73	14	19	41	63	58	9	46
22	44	57	71	3	49	30	76	17
24	43	56	70	2	51	29	78	16
64	5	54	32	81	10	27	37	59
35	75	13	21	40	62	67	8	48

Proposers' Solutions to Speed Problems

SD 1 *1.* $2^3 = 8$; *2.* $8 - 1 = 7$ (One placement when it is handed to you, seven moves to get to the remaining seven placements. 0, 1, 3, 2, 6, 7, 5, 4.); *3.* Three — any one of the three switches could be manipulated; *4.* $8 \times 3 = 24$ (Eight placements, three moves from each placement. 0, 1, 3, 7, 3, 2, 0, 2, 6, 7, 6, 4, 0, 4, 5, 7, 5, 1, 5, 4, 6, 2, 3, 1, 0); *5.* $8 \times 3 \times 3 = 72$; *6.* $72 + 1 = 73$ (0, 1, 3, 7, 6, 2, 0, 4, 5, 7, 3, 1, 0, 2, 6, 7, 5, 4, 0, 1, 5, 1, 5, 7, 5, 7, 6, 7, 6, 4, 6, 4, 0, 4, 0, 2, 0, 2, 3, 7, 3, 7, 5, 1, 0, 4, 6, 7, 3, 2, 6, 2, 6, 4, 5, 4, 5, 1, 3, 2, 3, 1, 3, 1, 5, 4, 6, 2, 3, 2, 0, 1, 0, 1).

SD 2 Straightforward algebraic derivations give (i) $x^2 - y^2 = (x - y)(x + y)$; and (ii) S must be divisible by 6.

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