Euclid Asks a Tough Question

Puzzle Corner
by
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Some sad news. Our trusty Fiat, which took us across the country several times on *very little* gas, has met its final reward. Typical story: The car in front stopped so I stopped, but the driver behind did not follow the same algorithm. It is annoying to find how much of a car can be untouched and still have the car considered a total loss.

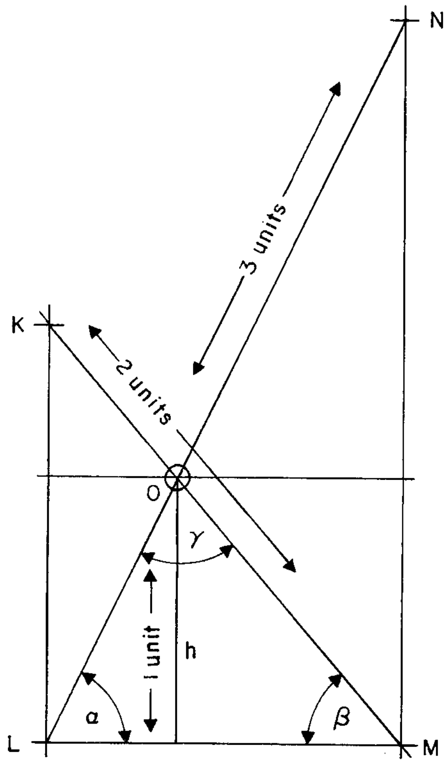
And an apology to the proposers of problems published as **JAN 3**, **FEB 3**, and **FEB SD 2**. Due to a clerical error, their names were misplaced. This is the ninth year for Puzzle Corner (eight in the *Review*), and I believe that this has never happened before. I am *sure* it has never happened in consecutive issues, and I hope to go another nine years before it happens again. Of course I will print the proposers' names should they care to identify themselves now.

Problems

M/A 1 We begin this month with a chess problem from Frank Rubin. These problems are in short supply. The problem: Black and White are to cooperate to checkmate White in the fewest possible moves, starting from the standard beginning position. What are the moves if Black is constrained to move only one pawn and White to move one pawn and one other piece only once?

M/A 2 This geometry problem, from Eric Jamin, is often called Feuerbach's Theorem. It begins with the nine-point theorem which was published as O/N 3 in 1973: Consider a triangle; prove that the three midpoints of the sides, the three basepoints of the altitudes, and the three midpoints of the segments joining the vertices to the orthocenter (the common intersection of the three altitudes) all lie on one circle. Now Mr. Jamin wants you to show that the "nine-point circle" defined above is tangent to the inscribed circle and to the exinscribed circles.

M/A 3 The following is from Walter G. Walker, who wants you to determine the sides and angles of the triangle LMO (uniquely defined) in the drawing. The two parallel lines KL and MN are perpendicular to the base LM. The height (h) is 1 unit, KM is 2 units, and LN is 3 units. The problem is to find the lengths of LO, OM, and LM and the angles α, β, and γ, and to show the methods and proofs used.



M/A 4 D. J. Huntley wants you to devise a simple scheme for deciding if a binary number (i. e., a number expressed in base 2) is divisible by 3.

M/A 5 R. E. Crandall suggests that we look for three distinct positive integers such that the sum of any two is a square.

Speed Department

SD 1 Our first "speed" problem is a "Backall Original," courtesy of F. Steele Backall: An inveterate jogger whose regular pace was consistently one mile in eight minutes noted that in the heavy traffic hours the flow of traffic was very steady, as to both the interval between cars and their speed. He also noted that there were more cars travelling in a northerly direction than in a southerly direction. One morning curiosity brought him to count the cars passing him as he jogged along his stretch of road. As he headed south from his home, he was interested to see that four times as many cars passed him headed north as headed south. When he turned around and headed north back to his home, he was startled to find that an equal number of cars passed him going in both directions. What was the speed of the traffic in miles per hour?

SD 2 Gary A. Ford says that an integer m contains an integer n if the digits of n form a subset of the digits of m. For example, 123 contains 1, 2, 3, 12, 13, 21, 23, 31, 32, 123, 132, 213, 231, 312, and 321. He asks, What is the smallest four-digit integer containing exactly five consecutive powers of some integer Q > 1?

Solutions

The following are solutions to problems published in the December, 1974, issue.

DEC 1 With the following hands, South has a seven-hearts-doubled contract and West leads the ♠9. Can South make his contract?

	♠ A Q 10 7	
	♥ 9 7 6 5 4 3	
	♦ 2	
	♣ Q 7	
♠ 9 8 5 4		♠ K J 6 3
♥ J		♥ 10
♦ K Q 4 3		♦ 9 7 6 5
♣ K J 9 8		♣ 5 4 3 2
	♠ 2	
	♥ A K Q 8 2	
	♦ A J 10 8	
	♣ A 10 6	

The following is from Edward Gershuny: South can make the contract. He plays the first ten tricks as follows:

1. Win with the ♠A.
2. Lead ♠10. East must cover with the ♠J and South trumps with an honor.
3. Lead a high trump to pull the defenders' teeth.
4. Lead the ♥2 to the board's ♥4.
5. Lead the ♠Q. East must cover with the ♠K and again South trumps high.
6. Cash the ♣A.
7. Lead the ♥8 to the board's ♥9.

8, 9, and 10. Play three rounds of trump. South discards two clubs and a diamond.

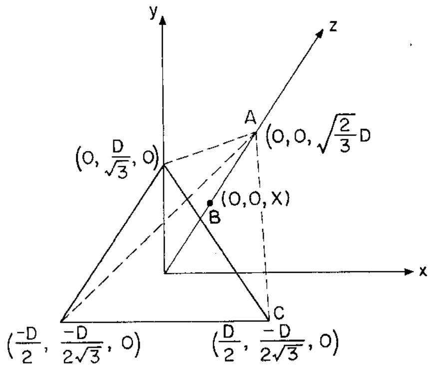
At this point, dummy holds the ♠7, ♦2, and ♣Q; and South holds ♦A, ♦J, and ♦10. East's holding is irrelevant. After the ninth trick, West held the ♠8 (to guard against dummy's ♠7), ♦K, ♦Q, and ♣K. He is squeezed by the last round of trump. West cannot discard a diamond because that would allow South to win the last three tricks in his hand. If West discards the ♠8, declarer will cash dummy's ♠7 (discarding the ♦10 from his hand) and West will have to discard either a diamond (allowing declarer to win the ♦A and ♦J) or his ♣K (allowing dummy's ♣Q and South's ♦A to win the last two tricks). If, at the tenth trick, West discards his ♣K, declarer will cash dummy's ♣Q forcing West to discard either a diamond (letting South win the last two tricks) or his spade (letting dummy's ♠7 and South's ♦A win the last two tricks).

Also solved by Mark Beach, Wallace Burton, Harvey Goldman, George A. Holderness, Robert Mandl, Avi Ornstein, R. Robinson Rowe, and the proposer, Michael Kay.

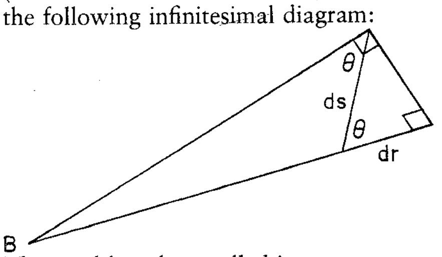
DEC 2 Four hovering hummingbirds are equidistant from each other when the first sees the second, the second sees the third, the third sees the fourth, and the fourth sees the first. Simultaneously each begins flying toward the one it sees, always pointing directly at it; they all fly at the same speed. How far will each travel before there is a collision?

The following solution is from Harvey Goldman:

Let D be the initial distance between the birds. The four birds may be assumed to start at the vertices of the following tetrahedron:



By symmetry, the impact point is at B (where x is to be determined). Consider the following infinitesimal diagram:



The total length travelled is

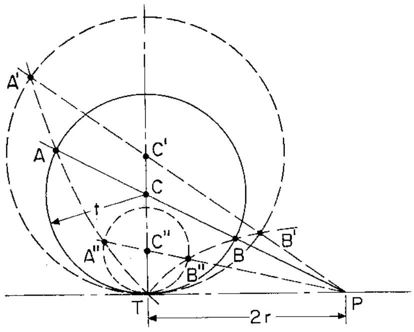
$\int ds = \int_0^r dr/\cos\Theta = R/\cos\Theta$ (note that $\cos\Theta = dr/ds$).

Since FB = AB, $[\sqrt{2/3}(D) - x]^2 = D^2/3 + x^2$, and $x = \sqrt{2/3}(D)/4$. Hence $R = \sqrt{2/3}(D) - x = 3\sqrt{2/3}(D)/4$.

Consider triangle ABC. Using the law of cosines and the coordinates of A, B, and C, one easily finds that $\cos\Theta = \sqrt{2/3}$. Thus the length travelled is $R/\cos\Theta = 3D/4$.

Also solved by R. Robinson Rowe.

DEC 3 In the diagram, PT is tangent to the circle and we are given that AB = PT. Prove that $(AP)(BP) = (AB)^2$.



The following interesting letter came from Egypt via Colorado:

"Sir:

"You will find the proof required in my *Elements*, Book III, Proposition 37.

"Have you made no progress in 2,250 years?

"Greetings,

"Euclid of Alexandria

"P.S. Above letter forged by Petr Beckmann."

I am also printing the following solution sent in by K. E. Schoenherr: We are given three conditions:

1. TP is tangent to the circle at T.
2. A slant line drawn from P intersects the circle at A and B.
3. The intercept AB is equal to a chosen interval TP on the tangent line.

The given conditions do not state the size of angle γ; therefore, the slant line may be drawn so that it passes through the center C of the circle. The intercept AB will then be 2r, where r is the radius of the circle. For this special case, AB = TP = 2r, whence the angle $\gamma = \tan^{-1}\frac{1}{2} = 26.6°$. Furthermore, for the right triangle CPT we have

$(r + b)^2 = r^2 + (2r)^2$

where b is the interval BP; so

$(2r + b)(b) = (2r)^2$, whence

$(AP)(BP) = (AB)^2$, which proves the proposition for this special case. In order to show that this special case is also the general case, draw two additional circles with centers on the normal to the line TP; these circles are shown in the figure by broken lines. Holding TP fixed, draw the slant lines through the centers of these circles and connect the points of intersection A′, A, A″ and B′, B, and B″ by smooth curves. It appears from the figure that the intercepts AB increase monotonously with increasing r and angle γ. Therefore, there is only one value of r for which the given condition (3) is satisfied. It follows that the special case for which the proposition was proved is also the general case.

Also solved by Gerald Blum, Robert Finney, Harvey Goldman, E. R. Leroy, Mary Lindenberg, Robert Mandl, Waller Moore, Avi Ornstein, John Prussing, R. Robinson Rowe, Victor Sauer, Harry Zaremba, and the proposer, Joseph Horton.

DEC 4 Prove or disprove the following: Given that $A^2 + B^2$, that A, B, and C are integers, and that C is not prime; then, for some factor F, of C, there exist integers G and H such that $G^2 + H^2 = F^2$.

Richard Bradley has supplied the following proof along with a method that could be used to derive Fermat's prime number theorem from this problem; due to space considerations, however, I'm forced to omit the latter. Here is the proof: The following Lemma is used: If A, B, p, and w are positive integers, $A^2 + B^2 = p \cdot w$, p is prime and $p \equiv 1 \bmod 4$, then there exist integers Q, R such that $Q^2 + R^2 = w$. The proof of the Lemma is that, from Fermat's theorem for prime numbers, there exist positive integers K, L for which $K^2 + L^2 = p$; then

$$(AK + BL)(AL + BK) = A^2KL + ABK^2 + ABL^2 + B^2KL$$
$$(AK + BL)(AL + BK) = (A^2 + B^2)KL + AB(K^2 + L^2)$$
$$A^2 + B^2 \equiv 0 \bmod p$$
$$L^2 + K^2 \equiv 0 \bmod p$$
$$(AK + BL)(AL + BK) \equiv 0 \bmod p$$
$$AK + BL \equiv 0 \bmod p \textit{ or } AL + BK \equiv 0 \bmod p.$$

Assume $AK + BL \equiv 0 \bmod p$ (if $AL + BK \equiv 0 \bmod p$, apply the following proof with K and L reversed); then

$$AK^2 + BKL \equiv 0 \bmod p$$
$$A(K^2 + L^2) \equiv 0 \bmod p$$
$$AL^2 - BKL \equiv 0 \bmod p \text{ (subtracting equations)}$$
$$0 < L < P$$
$$AL - BK \equiv 0 \bmod p.$$

Let

$$Q = (AK + BL)(1/p) \text{ and}$$
$$R = (AL - BK)(1/p).$$

Q and R are integers. Then

$$Q^2 + R^2 = (1/p)^2 (A^2K^2 + 2AKBL + B^2L^2 + A^2L^2 - 2AKBL + B^2K^2)$$
$$Q^2 + R^2 = (1/p^2 [(A^2 + B^2)(K^2 + L^2)]$$
$$Q^2 + R^2 = (1/p)^2(p \cdot w)p = w. \text{ Q.E.D.}$$

Now the original statement will be proved. In *case 1* C has a prime factor p such that $p \equiv 1 \bmod 4$.

$$A^2 + B^2 = p^2(c/p)^2.$$

There exist integers A_1, B_1 such that

$A_1^2 + B_1^2 = p(C/p)^2$ (applying the Lemma);

and there exist integers G, H such that

$G^2 + H^2 = p(C/p)^2$ (again applying the Lemma).

In *case 2* C is even. For any integer N, $N^2 \equiv 0$ or $1 \bmod 4$: $C^2 \equiv 0 \bmod 4$, $A^2 \equiv 0 \bmod 4$, and $B^2 \equiv 0 \bmod 4$. A and B are even. And $(A/2)^2 + (B/2)^2 = (C/2)^2$.

In *case 3* C has a prime factor p such that $p \equiv 3 \mod 4$. Either A and B are both multiples of p, or neither is. Suppose A and B are *not* multiples of p. The numbers $1,2,3, \ldots p-1$ are an abelian group under the operation of multiplication mod p. Call this group G. Let a, b be the elements in G for which $A \equiv a \mod p$ and $B \equiv b \mod p$. Then

$$a^2 + b^2 \equiv 0 \mod p$$

$$a^2 \equiv -b^2 \mod p.$$

Let a^{-1} denote the inverse of a in G. Then $(aa)^{-1} = 1$.

$$a^2(a^{-1})^2 \equiv -b^2(a^{-1})^2 \mod p$$

$$1 \equiv -(ba^{-1})^2 \mod p$$

$$(ba^{-1})^2 \equiv -1 \mod p.$$

The elements ba^{-1}, $p-1$, $(ba^{-1})(p-1)$, and 1 form a subgroup of G which has four elements. Therefore, the number of elements in G must be a multiple of four (Lagrange's Theorem in Group Theory). But G has $(p-1)$ elements and $p-1 \equiv 2 \mod 4$.

This contradiction renders the hypothesis invalid. A and B are multiples of p, and $(A/p)^2 + (B/p)^2 = (C/p)^2$. Every C must fit at least one of cases 1, 2, and 3, so the proof is complete.

Responses were also received from Gerald Blum, Avi Ornstein, R. Robinson Rowe, and Victor Sauer.

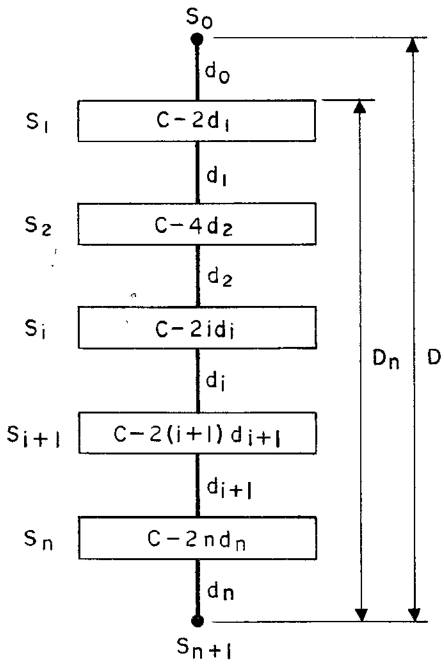
DEC 5 What strategy minimizes the amount of gas consumed (= miles travelled) traversing a 1,000-mile flat desert by car? The car gets 10 m.p.g.; its tank holds 25 gal. of fuel, and fuel may be carried in no other way; fuel may be left at any point along the way; there is an unlimited supply of fuel at mile 0; but, except for what is cached, gas is not available anywhere along the way.

Most answers agree to 3 or 4 decimal places. The following is from Harry Zaremba:

In the diagram shown on the next page,
S_0 = destination point
S_{n+1} = departure point
S_i = any fuel station established en route $(i = 1, 2, \ldots n)$
d_i = distance between any two stations
D_n = distance from departure point to the last established fuel station
D = total distance to destination
C = car's tank capacity in equivalent miles

Each expression in the box at each station is equal to the fuel which remains at that station before the final trip from S_{n+1} to S_0. The derivation of each expression is predicated on the following strategies to effect the most efficient and minimum consumption of gasoline:

1. Fuel tank of car is always maintained full before leaving any station S_{i+1} for S_i $(i = 0, 1, \ldots, n)$.
2. Fuel tank is permitted to contain only the amount of fuel necessary to reach station S_{i+1} when departing S_i $(i = 1, 2, \ldots, n)$.



In general, as a result of (1) the fuel consumed over distance d_i to reach S_i must be replenished at S_i to maintain a full tank; thus, on the final trip to S_0,

$$C - d_i + (C - 2id_i) = C,$$

or

$$d_i = C/(2i + 1),$$

and

$$D = d_0 + D_n = d_0 + \sum_{i=1}^{n} d_i = d_0 + C\sum_{i=1}^{n} 1/(2i + 1)\,.$$

The fuel represented by D_n in miles will be completely consumed and that corresponding to $d_0 = C$ will also be depleted if the tank is empty on arrival at S_0. With an arbitrarily selected D = 1,000 miles, and C = 250 miles, then $d_0 \leq 250$ and

$$D_n = 250\sum_{i=1}^{n} 1/(2i + 1) \geq 750\,.$$

Hence

$$\sum_{i=1}^{n} 1/(2i + 1) \geq 3\,,$$

which is satisfied when n = 418, resulting in

$$\sum_{i=1}^{418} 1/(2i + 1) = 3.000690592\,.$$

Consequently, $D_n = 250 \times 3.000690592 = 750.172648$ miles, and $d_0 = D - D_n = 249.827 - 352$ miles; and the car arrives at S_0 with $0.1(C - d_0) = 0.0172648$ gal. to spare. The total fuel consumed is equal to the number of times the tank was filled at the departure point less the amount remaining in the tank on arrival at S_0. If G is the total fuel consumed,

$$G = 0.1\,(n + 1)C - (C - d_0)$$

$$G = 0.1(nC + d_0) = 10474.982735 \text{ gal.}$$

Also solved by Edward Gershuny, Harvey Goldman, Neil Hopkins, Richard Kimble, Jr., R. Robinson Rowe, and Allen Wiegner.

Better Late Than Never

Responses to the problems indicated have been received:

O/N1 Gerald Blum and Paul Reeves
O/N2 Gerald Blum, Avi Ornstein, and Paul Reeves
O/N3 Gerald Blum and Charles Foster
O/N4 Gerald Blum, William Blum, Leonard Charek, Robert A. Keller, Michael Kotch, Paul Reeves, and John Welch.

Proposers' Solutions to Speed Problems

SD1 22.5 m.p.h.
SD2 1256 contains 1, 5, 25, 125, and 625.

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