Beginning the First Annual Problem

Puzzle Corner
by
Allan J. Gottlieb

Rugby season is over, and I survived. I lost two toenails and sustained a couple of annoying muscle pulls—but all in all escaped essentially unscathed.

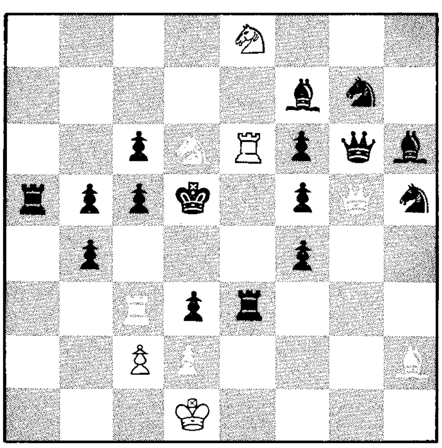
As promised a few issues ago, I shall follow the suggestion of Roger Lustig and institute a yearly problem based on PERM 1. This problem will run the entire year of 1975 (reprinted in later issues), and the best answers will be published next January; here goes:

Y1975 From the four digits 1, 9, 7, and 5, construct integers from 1 to 100 (none higher will be published) using *only* the following symbols:
+, −, *, /, **, (, and).
** denotes exponentiation (and is easy to typeset). Digits may be juxtaposed, and the best answer for a given number is the one with the lowest "point value." One point is given for each occurrence of +, −, *, /, or **. For example, 1 = 1**975 gets one point; 2 = − 1 − 9 + 7 + 5 gets four points; and 2 = 7 + 5 − 1 − 9 gets three points. Thus the third example is better than the second even though the digits are out of order. Remember that solutions above 100 or using any other symbols will not be printed.

Problems

We start this month with a chess problem (chess problems, by the way, are in short supply) from Harry Nelson:

JAN1 White to play to win:



JAN2 Les Servi wants you to prove that among triangles of a given perimeter the equilateral has maximal area.

JAN3 A very interesting astronomy problem has come from a reader whose identity has been lost (apologies!): We have seen Kohoutek as a nearly parabolic comet. What is the maximum time a truly parabolic comet can remain inside the earth's orbit?

JAN4 Paul de Vegvar would like you to find all x such that $x^x = i$.

JAN5 Frank Rubin wonders if it is possible to have a magic square with each entry prime. Can you make each entry a distinct prime?

Speed Problems

SD1 Here is a bio problem from J. Keilin: Assume an amoeba reproduces itself every three minutes. If a jar having one amoeba is filled with them in an hour, how long would it take starting with two amoebas?

SD2 The following is from Mary Lindenberg: A game is played by three men with the understanding that the loser is to double the money of the other two. After three games, each has lost just once, and each has $24. How much did each man have at the start of the game?

Solutions

The following are solutions to problems published in the July/August, 1974, issue:

J/A1 A scrambled sequence of letters is formed by choosing a keyword (containing no repeated letters) and writing the remainder of the alphabet in order after it. The letters of this sequence are written as capitals on the 26 cards ♠A to ♥2. The letters of another sequence (based on a different keyword) are written as small letters on the 26 cards ♦A to ♣2. A bridge game, contract 4♥ by S., down one, might be written as shown below.

```
W N E S W N E S W N E S W N E S W N E S W N E
d s c g
  G D R O
        p z n m
              H X V Y
              T E C M
                    h w u q
                    x k N l
                        y o e a
                          S U J L
                              F Z P B
                                  f r W i
                                     K b t v
                                     Q A j l
```

What are the keywords?

The following is from Winslow H. Hartford:

	A	K	Q	J	10	9	8	7	6	5	4	3	2
♠	*A*	*M*	*O*	*R*	*T*	*I*	*Z*	*E*	B	E	D	F	G
♥	H	J	K	L	N	P	Q	S	U	V	W	X	Y
♦	*s*	*o*	*b*	*r*	*i*	*e*	*t*	*y*	a	c	d	f	g
♣	h	j	k	l	m	n	p	q	u	v	w	x	z

The keywords are in italics. The original holdings were:

```
            ♠ 5 2
            ♥ 9 7 5
            ♦ A K 8 3
            ♣ K J 7 2
♠ A Q 8 7                ♠ K 9 6 4
♥ J 10 3                 ♥ 6 2
♦ Q 10 7 4               ♦ J 9 5
♣ 8 6                    ♣ A 9 5 3
            ♠ J 10 3
            ♥ A K Q 8 4
            ♦ 6 2
            ♣ Q 10 4
```

Also solved by Eric Jamin, R. Robinson Rowe, and the proposer, Walter F. Penny.

J/A2 For positive x, let $y_1 = x$, $y_2 = x^x$, and in general $y_n = x^{y_{n-1}}$. Now let $Z_n = \lim_{x\to 0} y_n$. In terms of n, what is Z_n?

Donald Aucamp answers that $Z_n = 0$ for n odd and $Z_n = 1$ for n even. His proof involves the axiom of induction. Trivially, $Z_1 = 0$.

$$Z_2 = \lim_{x\to 0^+} y_2 = \lim_{x\to 0^+} x^x \lim_{x\to 0^+} e^{x\ln(x)} = 1$$

The above result follows from the fact that, for $t = -\ln(x)$, $\lim_{x\to 0^+} x\ln(x) = \lim_{t\to\infty} -te^{-t} = 0$. Now, for sufficiently small x, $y_2 = 1 + x\ln(x) + 0_2 \geqq 1 + 2x\ln(x)$, where 0_2 is second order in $x\ln(x)$. This is true since, for small x, $0_2 \geqq 0$ and $x\ln(x) < 0$. In invoking the axiom of induction, we assume $Z_{2n-1} = 0$, $Z_{2n} = 1$, and $y_{2n} \geqq 1 + 2x\ln(x)$ for small x. These assumptions have been shown above to be valid for n = 1. If $Z_{2n+1} = 0$, $Z_{2n+2} = 1$, and $y_{2n+2} \geqq 1 + 2x\ln(x)$, the proof is complete by induction.

$$Z_{2n+1} = \lim_{x\to 0^+} x^{y_{2n}} = (0)^{(1)} = 0$$

$$y_{2n+2} = x^{(x^{y_{2n}})}$$

$$\ln y_{2n+2} = x^{y_{2n}}\ln(x).$$

Since $x^A \geqq x^B$ when $0 < A < B$ for $0 \leqq x \leqq 1$, then $x^{1+2x\ln(x)} \geqq x^{y_{2n}}$ and $x^{1+2x\ln(x)} \leqq x^{y_{2n}}\ln(x) = \ln y_{2n+2}$. Thus, since $x\ln(x) \to 0$ and x^A is bounded by [0, 1] for $0 \leqq x \leqq 1$ for all A,

$$\lim_{x\to 0^+} \ln y_{2n+2} \geqq \lim_{x\to 0^+} x^{1+2x\ln(x)}\ln(x) = \lim_{x\to 0^+} x\ln(x)\, x^{2x\ln(x)} = 0. \text{ Thus}$$

$Z_{2n+2} \lim_{x \to 0^+} e^{\ln y_{2n+2}} = e^0 = 1$. We now conclude the proof by showing $y_{2n+2} \geqq 1 + 2x \ln(x)$ for x sufficiently small. $x^{y_{2n-1}} \leqq 1 < 2$. Thus $x^{y_{2n}} \leqq 2x$. Thus $x^{y_{2n}} \ln(x) \geqq 2x \ln(x) \geqq \ln(1 + 2x \ln x)$. Thus,

$e^{x^{y_{2n}} \ln x} \geqq e^{\ln(1+2x \ln x)} = 1 + 2x \ln(x)$.

But

$e^{x^{y_{2n}} \ln x} = e^{\ln x^{x^{y_{2n}}}} = x^{x^{y_{2n}}} = y_{2n+2}$.

Therefore, $y_{2n+2} \geqq 1 + 2x \ln(x)$, and the proof is complete.

Also solved by Joseph Horton, Eric Jamin, John Prussing, R. Robinson Rowe, Frank Rubin and the proposer, Neil Judell.

J/A3 What is the probability of a successful blood transfusion (one in which no adverse reaction occurred among major factors)? You need to know:

		Donor			
	Type	**A**	**O**	**B**	**AB**
Recip-ient	**A**	ok	ok	x	x
	O	x	ok	x	x
	B	x	ok	ok	x
	AB	ok	ok	ok	ok

		Donor	
	rh	**+**	**—**
Recip-ient	+	ok	ok
	—	x	ok

The distribution of blood types in the population is approximately as follows: A—40 per cent, O—40 per cent, B—15 per cent; and AB—5 per cent; rh+ —85 per cent; and rh— —15 per cent. Types and rh's are randomly mixed—that is, 85 per cent of each type is rh+ and 15 per cent is rh—.

The following is from Jim Toker: The first step in solving this problem is to calculate the percentage of people having each combination of rh factor and blood type:

		rh factor	
		+	**—**
Blood type	**O**	34	6
	A	34	6
	B	12.75	2.25
	AB	4.25	0.75

Next, set up a table of all the various combinations of donor and recipient blood types, and eliminate those combinations in which the transfusion fails (top of cols. 2-3). Then, using the percentages from the first table, calculate the probability of occurrence for each of the remaining combinations. Finally, add up the probabilities of each successful transfusion to obtain the solution—that the probability of having a successful transfusion is 53.440625 per cent.

			Donor							
	Type		**O**		**A**		**B**		**AB**	
		rh	+	—	+	—	+	—	+	—
Recip-ient	**O**	+	11.56	2.04	x	x	x	x	x	x
		—	x	0.36	x	x	x	x	x	x
	A	+	11.56	2.04	11.56	2.04	x	x	x	x
		—	x	0.36	x	0.36	x	x	x	x
	B	+	4.335	0.756	x	x	1.625625	0.286875	x	x
		—	x	0.135	x	x	x	0.050625	x	x
	AB	+	1.445	0.255	1.445	0.255	0.541875	0.095625	0.180625	0.031875
		—	x	0.045	x	0.045	x	0.016875	x	0.005625

Also solved by Winslow Hartford, Neil Hopkins, Eric Jamin, R. Robinson Rowe, Frank Rubin, Harry Zaremba, and the proposer, Joseph Horton.

J/A4 A standard deck of 52 cards is shuffled and placed face down upon the table. The cards are then turned face up one at a time by flipping over the top card of the face-down stack. As this is done, the player simultaneously calls out the sequence A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A, 2, etc., one call being made for each card flipped over. To win the game, one must go through the deck without matching a card flipped over with the card called. Suits do not matter; for example, any four-spot flipped over on the 4th, 17th, 30th, or 43rd turn results in a loss. What are the chances of winning the game? How about a second solution for the same game with a 48-card pinochle deck?

For this one everyone gave approximations; the following is from Harry Zaremba: The derivation and manual computation of an exact solution would be a formidable task. However, a good approximate answer is given by the formula:

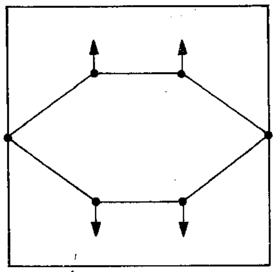
$$P_o = e^{-k}\left[1 - \frac{k(k-1)}{2(n-1)} + \frac{k(k-1)(3k^2 - 11k + 16)}{24n(n-1)}\right]$$

where P_o is the probability of the event that no card matches a call, n is the number of cards in the deck, and k is the number of identical cards in each of m different sets of cards in the deck in which the suits are ignored ($n = mk$). Thus, for a 52-card deck, $n = 52$, $m = 13$, and $k = 4$; therefore $P_o = 0.0162299$, or about one chance in 62 trials that the game could be won. In a 48-card pinochle deck, $n = 48$, $m = 6$, and $k = 8$; $P_o = 0.0001772479$, or one chance in approximately 5,642 trials for a game to be won. Generally, when n and $m \to \infty$, P_o approaches e^{-k}. I have computed an exact probability value for $n = 12$, $m = 3$, and $k = 4$, which is $P_o = 0.00998557$. The formula yields $P_o = 0.0097128$, which is a fair approximation even for a low magnitude of n.

Also solved by Neil Cohen, Winslow Hartford, Neil Hopkins, Joseph Horton, Eric Jamin, Judith Q. Longyear, Jack Parsons, Craig Presson, R. Robinson Rowe, and Frank Rubin.

J/A5 Place six points inside a unit square so that the nearest two are as far apart as possible.

The only proof supplied is from the proposer, Frank Rubin, whose solution follows: Like many innocent-looking problems, this one has hidden teeth. Clearly, at least three of the points must be on the boundary or we could simply spread the remainder apart (Figs. 1 and 2 at the top of the next page). Assume three interior points; if they are in a line (Fig. 3), the farthest they can be separated is along the diagonal (Fig. 4). This configuration obviously cannot be better than that of Fig. 10. If the three interior points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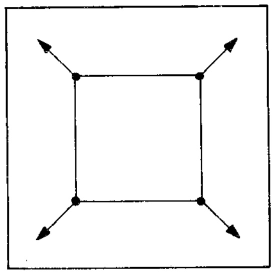
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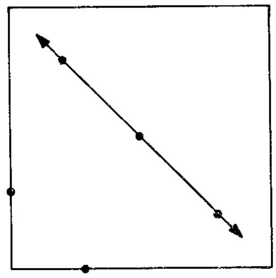
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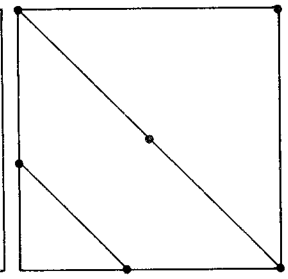
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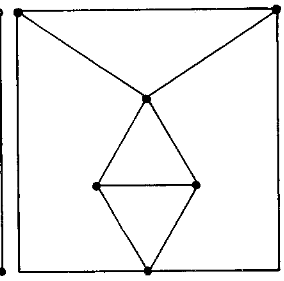
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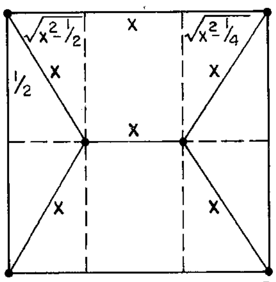

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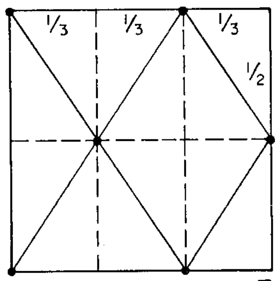

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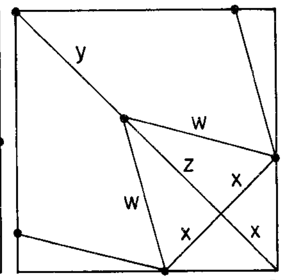

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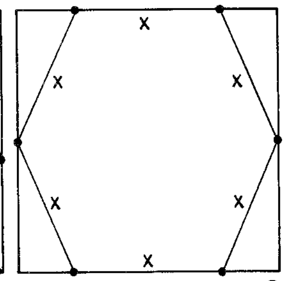

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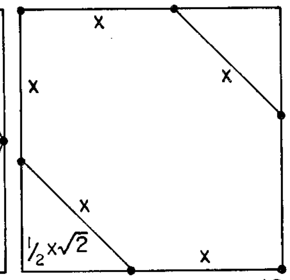

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form a triangle, the best configuration is shown in Fig. 5. Clearly this cannot be better than Fig. 7. So assume two interior points. The only reasonable configuration seems to be that of Fig. 6. Here $x + 2\sqrt{x^2 - .25} = 1$, so $x = (\sqrt{7} - 1)/3 = 0.5485$. Next, assume one interior point. There are two reasonable configurations, Figs. 7 and 8. In Fig. 7, $x = \sqrt{1/4 + 1/9} = \sqrt{13}/6 = 0.6009$. In Fig. 8 we can show that the minimum distance is less than 0.6; for if $x \geqq 0.3$ and $y \geqq 0.6$, then Z can be at most $\sqrt{2} - 0.9 = 0.514$, so $w \leqq \sqrt{0.514^2 + 0.3^2} = \sqrt{0.352}$, which is less than 0.6. Finally, consider the possibility of no interior points whatever. The reasonable configurations are Figs. 9 and 10. Fig. 9 obviously has x the same as Fig. 6, namely $x = 0.5485$. In Fig. 10 we have $x + \frac{1}{2} \cdot \sqrt{2} = 1$, $x = 2 - \sqrt{2} = 0.5858$. The optimal configuration is therefore Fig. 7, with $x = 0.600925$.

Responses were also received from Walter Daugherity, Joseph Horton, Eric Jamin, Winslow Hartford, and Harry Zaremba.

Allan J. Gottlieb studied mathematics at M.I.T. (S.B. 1967) and Brandeis (A.M. 1968, Ph.D. 1973) he is now Assistant Professor of Mathematics at York College of C.U.N.Y. Send problems, solutions, and comments to him at the Department of Mathematics, York College, 150-14 Jamaica Ave., Jamaica, N.Y. 11432.

Letters

Continued from p. 4

while, manure from feed lots has become a major solid waste disposal problem. Granted that manure (and much compost) is deficient in valuable trace minerals, modest applications of lime, bone meal, wood ashes, or powdered rock will usually correct the balance. The wise organic farmer tests the soil and develops its fertility over a period of years.
William F. Hoey
Hamden, Conn.

Saving Energy: Apples vs. Oranges, and Consider the Lowly Motorcycle

The chart accompanying your report on "Conserving Transport Energy" (*"Trend of Affairs," June, pp. 54-55*) indicates that converting to 50 per cent small cars would result in energy savings of 9 per cent and converting 50 per cent of cars to 30 per cent better mileage would result in savings of 10.25 per cent. The Department of Transportation says the average mileage now is 13 m.p.g.; a 30 per-cent improvement would be 17 m.p.g. Assuming a random distribution in both conversions, I find it hard to believe that you believe that 17 m.p.g. is saving more than 22 m.p.g.

The 1 to 2 per cent savings by walking is almost insignificant—except that it represents close to 100,000 bbl./day of oil, or the entire output of a fair-size refinery. It also represents the mentality of the designers and engineers who have locked the U.S.A. into an inflexible system that cannot be changed except by tremendous capital expenditures over several generations.
Charles Donaho
Houston, Texas

Since the article listed the assumptions Dr. Malliaris used, I proceeded to check the calculation that led to his estimate that a 50 per-cent conversion to small cars will produce fuel savings of only 9 per cent. He assumed that 90 per cent of today's cars get 13.1 m.p.g. and 10 per cent (the small cars) get 22 m.p.g. He then assumed an increase in the small car population to 50 per cent. My numbers, based on a 100-mile trip, are as follows: Present situation—50 miles at 13.1 m.p.g. = 3.82 gal. and 50 miles at 22 m.p.g. = 2.27 gal. for a total of 6.09 gal. or 17 per cent less than before.

Dr. Malliaris makes no mention of motorcycle use in urban situations to conserve fuel. Small motorcycles have been ignored for too long as fuel-saving possibilities. I propose that, instead of converting 50 per cent of the vehicle population to small cars, we convert 50 per cent to small motorcycles averaging, say, 75 m.p.g. Assume the motorcycles are used only two-thirds of the time—allowing for rain and cold weather. The remaining vehicle population could be 10 per cent large cars at 13.1 m.p.g. and 40 per cent small cars at 25 m.p.g. Thus, two-thirds of the time, with 50 per cent motorcycles, 40 per cent small cars, and 10 per cent large cars, we need 3.03 gal.; one-third of the time, with 80 per cent small cars and 20 per cent large cars, we need 4.73 gal., for an average of 3.6 gal. or a savings of 51 per cent.

Some may consider my estimates a bit too optimistic. But the point is that there is a vast potential for fuel savings in the urban situation without requiring substantial changes in life styles.
D. Thomas Terwilliger
Lafayette, Ind.

Dr. A. C. Malliaris of the Transportation Systems Center, Cambridge, whose report was summarized in Technology Review, *clarifies our report as follows*:

If one wishes to avoid comparing apples with oranges, then the conservation potential of each individual action must be determined as a percentage of the total transportation energy. . . . I repeat here two elementary calculations for the benefit of the above and any other unhappy readers: To convert 50 per cent of the passenger car population to small cars (22 m.p.g.) we convert from a population in which 90 per cent of the cars achieve 13.1 m.p.g. and 10 per cent achieve 22 m.p.g. (an average of 13.65 m.p.g.) to one in which half the cars achieve 13.1 m.p.g. and half 22 m.p.g. (an average of 16.42 m.p.g.) This is a 16.9 per cent reduction in the fuel consumption of passenger cars, or a reduction of 9 per cent in the total of transportation energy, since passenger cars account for about 53.5 per cent of the total transportation energy.

Consider now the next action: introduce in 50 per cent of highway vehicles a 30 per cent reduction in fuel consumption. Highway vehicles include not only passenger cars but also trucks, buses, etc.; together these vehicles account for 76.5 per cent of the total transportation energy. Obviously, the action under consideration here would yield 50 per cent × 30 per