South from the South Pole?

Puzzle Corner
by
Allan J. Gottlieb

I read in the *New York Times* that Larry Kaufman of Baltimore, Md., is competing in a top class chess tournament and has the possibility of becoming an international master if he does well. Larry was a colleague of mine at M.I.T. and voted for me for Baker House hall chairman. I should like to return the favor and root for him in the tournament. I won the election, by the way, and I hope the analogy continues.

So that everyone understands my expertise at chess, however, let me point out that Rockefeller University has just held a speed chess tournament, and your editor finished sixth (out of eight, unfortunately) with a score of 1½-5½. Oh well, I can still play quarterback.

Jerry Stoodley, a colleague of mine at York, has suggested that **O/N3** be extended to a permanent problem. As you will see below, the integers 1 to 115 can be written using the digits 1, 9, 7, and 3, together with certain algebraic operations. The permanent problem is to raise the limit of 115. I will publish new results as they are achieved but, due to space considerations, will print only one way of writing each number.

Speed problems are still in short supply.

Problems

A bridge problem from Winslow H. Hartford:

FEB1 Find a way for South to make four hearts given a lead of ♠ K:

	♠ A 5	
	♥ 7 6 5 3 2	
	♦ K 7 2	
	♣ A 8 2	
♠ K Q 10 9 8 4 3		♠ 7 6
♥ Q J 8		♥ —
♦ 8 3		♦ Q J 10 9 5
♣ 10		♣ K Q J 7 6 3
	♠ J 2	
	♥ A K 10 9 4	
	♦ A 6 4	
	♣ 9 5 4	

R. E. Crandel wants you to:

FEB2 Find all the primes of the form $a^4 + 4b^4$.

This problem, from Allen Anderson, is to be solved using only geometry (no trigonometry or analytic geometry allowed):

FEB3 Let ABC be an equilateral triangle inscribed in a circle. Choose any point P on the arc from B to C. Prove that PA = PB + PC.

Jerome Miller wants you to:

FEB4 Find a seven-fold homonym in the English language (or six-fold, if proper nouns are excluded).

Here is a shipping problem from Smith D. Turner:

FEB5 The parcel post box size is limited by the total of length and girth (twice the width plus the depth) while air baggage is limited by the total of the three dimensions. Say, for example,

$P_M = L + 2W + 2D = 84$

$A_M = L + W + D = 62$

What is the largest volume box that can be shipped either way (a) generally, and (b) with P_M and A_M as above?

Speed Department

Although a quicky, this problem submitted by Jack Parsons stumped Cardan (of cubic equation fame) to the extent that, after proposing the problem, he gave the wrong solution. Read the problem carefully:

FEBSD1 What is the probability of obtaining an ace at least twice in three throws (of two dice)?

The following is from Tom Hastings:

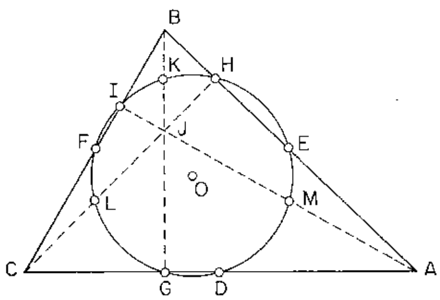
FEBSD2 There's an old kid's riddle about the hunter who travels one mile south, one mile east, and one mile north in such a way that he is back where he started from. He then shoots a bear. What color is the bear? The usual answer is white, the hunter having started and ended his excursion at the North Pole. Are there any other points on the globe from where the hunter could have started?

Solutions

The following solutions are to problems published in the October/November issue, omitting **O/N1**, which was modified in the January issue and whose solution will be given in the May issue, with the rest of the January solutions.

O/N2 Consider a triangle: Prove that the three midpoints of the sides, the three basepoints of the altitudes, and the three midpoints of the segments joining the vertices to the orthocenter (the common intersection of the three altitudes) all lie on one circle.

The following solution is from R. Robinson Rowe:



Given triangle ABC with median points D, E, and F; altitude feet at G, H, and I; orthocenter at J; and midpoints between J and vertices at K, L, and M; to prove that D, E, F, G, H, I, K, L, and M all lie on a circle. Use analytic geometry, first taking the origin of cartesians at C, line CA as the x-axis, and coordinates of the vertices generalized as C = (0,0), A = (2a,0), and B = (2b,2c). Then the equations of the lines are:

CA: $y = 0$

CB: $cx - by = 0$

BG: $x = 2b$

AI: $bx + cy = 2ab$

and coordinates of points are determined by averaging those of vertices to obtain median points and by intersecting lines to obtain I and J, and by averaging B and J to get K, listed below:

D: $(a,0)$

E: $(a + b,c)$

F: (b,c)

G: $(2b,0)$

I: $(2ab^2/[b^2 + c^2], 2abc/[b^2 + c^2])$

J: $(2b,2b(a - b)/c)$

K: $(2b,[c^2 + ab - b^2]/c)$

Now any three noncollinear points determine a circle, and I choose D, F, and G, substituting coordinates of these points in the general form:

$x^2 + y^2 + 2hx + 2iy + k = 0$

getting the three equations in h, i, and k:

$a^2 + 2ah + k = 0$

$b^2 + c^2 + 2bh + 2ci + k = 0$

$4b^2 + 4bh + k = 0$

whence

$2h = -(a + 2b)$

$2i = -(c^2 + ab - b^2)/c$

$k = 2ab$

and the equation of circle DFG is

$x^2 + y^2 - (a + 2b)x - (c^2 + ab - b^2)y/c + 2ab = 0$

Substituting the coordinates of E for x and y, that is, $x = a + b$ and $y = c$:

$a^2 + 2ab + b^2 + c^2 - a^2 - 3ab - 2b^2 - c^2 - ab + b^2 + 2ab = 0,$

which checks. Similarly with the coordinates of K, that is, $x = 2b$, $y = (c^2 + ab - b^2)/c$:

$4b^2 + (c^2 + ab - b^2)^2/c^2 - 2ab - 4b^2 - (c^2 + ab - b^2)^2/c^2 + 2ab = 0,$

which checks. Hence E and K lie on circle DFG. Next, take the origin of coordinates at A and line AB as the x-axis. The same general analysis would define a circle EHD embracing points F and L. Since circle DEFGK and circle DEFHL each contain three points D, E, and F, the circles coincide. Finally, take the origin at B and line BC as the x-axis. The same general analysis would find DEFIM on a circle—again the same circle. Thus all nine points D, E, F, G, H, I, K, L, and M lie on a circle. The center of this circle, incidentally, is located on the perpendicular bisectors of line segments DG, EH, FI, GK, HL, and IM, at point O on the diagram.

Also solved by Eric Jamin, Mary Lindenberg, and the proposer, Zachary Gilstein.

O/N3 Take the digits 1, 9, 7, and 3 and, using any mathematical symbols, construct an expression yielding each number from 1 to 10 (each of the four digits must be used exactly once); for example, 1^{973} yields 1.

As mentioned above, this problem is being extended to a permanent problem. Allowable operations are addition, subtraction, multiplication, division, factorial, and square roots. In addition, numbers may be raised to powers, digits may be juxtaposed (e.g., 17 is seventeen in case you didn't know), and brackets [ ] may be used to indicate the greatest integer not greater than the number within the brackets. You may use the 1, 9, 7, and/or 3 in conjunction with $\sqrt{\ }$ to get other roots.

Captain E. B. Jarman has gone far beyond the original proposer's demand by obtaining solutions up to 108, 110 through 115, and a few scattered numbers beyond. Your editor was delighted to find a solution to 109 which uses !! (factorial of the factorial). By the way, parentheses are allowed for indicating the order of operations. Here are Captain Jarman's solutions. Captain Jarman's solutions are in the box at the bottom of the page.

Forty-one others responded to this problem, but space limitations preclude our listing their names. Why not get to work on 116?

O/N4 Is it true (asked Winthrop M. Leeds) that no odd integers A and B satisfy $A^2 + B^2 = C^2$ (of course C would be even)?

Not to spoil any intended meter in Neil Hopkins' literary solution, I have refrained from changing "factor" to "multiple." Here is the solution:

Mr. Leeds has uttered an eternal verity. May his days be long and prosperous. He has had the insight to note that C^2, being even, requires that C also be even, of the form 2p where p is an integer. C^2 therefore equals $4p^2$ which is a factor of 4. The left-hand side of the equation, however, is downright stubborn and set in its ways. It will have nothing to do with this factor 4 business. It remembers that it is comprised of two terms, both of which are odd, requiring that A and B be of the form of $2m + 1$ and $2n + 1$, respectively, where m and n are integers. The left hand side of the equation then becomes $4(m^2 + n^2 + m + n) + 2$, a factor of 2 but not of 4. The number theory boys and girls will crack this little nubbin in 20 sec. flat. May their progeny thrive and multiply.

The following interesting note is from Philip A. Gagner:

Pascal, as you might know, used this very problem in his "Pensees" to demonstrate the difference between the kind of knowledge that involved the rational mind and the kind of knowledge that is "intuitive" to the heart. He stated that the concept of number was intuitive and from the heart, but that the proof of such a theorem was from the mind.

Responses were also received from Bob Baird, Don Cathcart, Theodore Edison, Paul Greenfeld, Winslow Hartford, Ned Horvath, Eric Jamin, Capt. E. B. Jarman, Winthrop Leeds, Henry Lieberman, Harry Nelson, Mitchell Raps, Adam Reed, Ben Rouben, R. Robinson Rowe, Frank Rubin, John Rule, Victor Sauer, Lawrence Smith, and Harry Zaremba.

O/N5 Evaluate exactly

$$x = \sqrt{121 + 15\sqrt{65}} - \sqrt{(11\sqrt{2} - 5\sqrt{5})(11\sqrt{2} - 3\sqrt{13})}$$

It is easy to get an approximate value for an answer. However, surprisingly enough, there is an exact integral answer. The following algebraic solution is due to Ben Rouben:

Let $x = \sqrt{121 + 15\sqrt{65}} - \sqrt{(11\sqrt{2} - 5\sqrt{5})(11\sqrt{2} - 3\sqrt{13})}$

Let $121 + 15\sqrt{65} = (a + b)^2 = a^2 + 2ab + b^2$. In fact, let

$$a^2 + b^2 = 121 \quad (1)$$

and let

$$2ab = 15\sqrt{65}. \quad (2)$$

Solve (2) for b and put this into (1), giving

$a^2 + (225)(65)/4a^2 = 121.$

So $4a^4 - 484a^2 + 14625 = 0.$

Using the quadratic formula we get $a^2 = 62\frac{1}{2}$ or $58\frac{1}{2}$, so $b^2 = 58\frac{1}{2}$ or $62\frac{1}{2}$. By (2) a and b have the same sign which we may as well assume is positive. Thus $a = 5\sqrt{2\frac{1}{2}}$ and $b = 3\sqrt{6\frac{1}{2}}$. So

$$\sqrt{121 + 15\sqrt{65}} = 5\sqrt{2\tfrac{1}{2}} + 3\sqrt{6\tfrac{1}{2}}. \quad (3)$$

Since $(5\sqrt{2\frac{1}{2}} + 3\sqrt{6\frac{1}{2}} - 11)^2 = (11\sqrt{2} - 5\sqrt{5})(11\sqrt{2} - 3\sqrt{13})$, we have

$$\sqrt{(11\sqrt{2} - 5\sqrt{5})(11\sqrt{2} - 3\sqrt{13})} = 5\sqrt{2\tfrac{1}{2}} + 3\sqrt{6\tfrac{1}{2}} - 11 \quad (4)$$

Plugging (3) and (4) into the definition of x shows that $x = 11$.

Responses were also received from Bob Baird, Paul Greenfeld, Winslow Hartford, Sheldon Hoffman, Arthur Hou, Eric Jamin, Harry Nelson, Frank Rubin, Victor Sauer, Peter Silverberg, Harry Zaremba, and the proposer, R. Robinson Rowe.

$1 = (1 + 9)/(7 + 3)$	$35 = 1 + 7 + (9 \cdot 3)$	$67 = 1 + (9 \cdot 7) + 3$	$99 = 97 - 3 - 1$
$2 = 7 \cdot 3 - 19$	$36 = (9 - 3) \cdot (7 - 1)$	$68 = 71 - 9/3$	$100 = 97 + (1 \cdot 3)$
$3 = 9 + 7 - 13$	$37 = 3 \cdot (1 + 9) + 7$	$69 = 9 \cdot (1 + 7) - 3$	$101 = 1 + 97 + 3$
$4 = (9 + 7)/(1 + 3)$	$38 = 7 \cdot 3! - 1 - \sqrt{9}$	$70 = 91 - (7 \cdot 3)$	$102 = 97 + 3! - 1$
$5 = 9 - 7^1 + 3$	$39 = 1^7 \cdot 39$	$71 = 71 \cdot \sqrt{9}/3$	$103 = 9!/7! + 3!$
$6 = 1 + 9 - 7 + 3$	$40 = (1 + \sqrt{9}) \cdot (7 + 3)$	$72 = 9 \cdot (7 + 1^3)$	$104 = 1 + 97 + 3!$
$7 = 1^{(9+3)} \cdot 7$	$41 = 7 \cdot (9 - 3) - 1$	$73 = (1 + 9) \cdot 7 + 3$	$105 = 7 \cdot 1 \cdot (9 + 3!)$
$8 = 1^{(9+3)} + 7$	$42 = 1 \cdot (9 - 3) \cdot 7$	$74 = 1^9 + 73$	$106 = 1 + 7 \cdot (9 + 3!)$
$9 = 19 - 7 - 3$	$43 = 1 + (9 - 3) \cdot 7$	$75 = (1 + 7) \cdot 9 + 3$	$107 = (19 \cdot 3!) - 7$
$10 = (1^9 \cdot 3) + 7$	$44 = 17 + (9 \cdot 3)$	$76 = 93 - 17$	$108 = (1 + 7) \cdot (9 + 3)$
$11 = (7 \cdot 3) - 9 - 1$	$45 = (1 + 7 - 3) \cdot 9$	$77 = 1 + 79 - 3$	$109 = [3!!/7] + \sqrt{9}! + 1$
$12 = 1^7 \cdot (9 + 3)$	$46 = 37 + (1 \cdot 9)$	$78 = 3 \cdot (17 + 9)$	$110 = 39 + 71$
$13 = 1^7 + 9 + 3$	$47 = (9 + 7) \cdot 3 - 1$	$79 = 1^3 \cdot 79$	$111 = 1 \cdot \sqrt{9} \cdot 37$
$14 = 17 - 9/3$	$48 = 79 - 31$	$80 = 1^3 + 79$	$112 = (7 \cdot 3) + 91$
$15 = 13 + 9 - 7$	$49 = 7^{(9/3-1)}$	$81 = 79 + 3 - 1$	$113 = (9 - 3 - 1)! - 7$
$16 = 91/7 + 3$	$50 = (19 \cdot 3) - 7$	$82 = (13 \cdot 7) - 9$	$114 = (1 + 37) \cdot \sqrt{9}$
$17 = 1^3 + 9 + 7$	$51 = 17 \cdot 9/3$	$83 = 1 + 79 + 3$	$115 = 91 + (7 - 3)!$
$18 = 91 - 73$	$52 = 7\sqrt{9} + 31$	$84 = 97 - 13$	116 = ✓
$19 = 7 + 3 + (1 \cdot 9)$	$53 = (9 - 1) \cdot 7 - 3$	$85 = 93 - 7 - 1$	117 = ✓
$20 = 1 + 9 + 7 + 3$	$54 = 73 - 19$	$86 = 19 \cdot (7 - 3)$	118 = ✓
$21 = 1 \cdot 9/3 \cdot 7$	$55 = 1^7 + (9 \cdot 3!)$	$87 = 1 + 93 - 7$	$119 = (9 - 7 + 3)! - 1$
$22 = 1 + (9/3 \cdot 7)$	$56 = 19 + 37$	$88 = (13 \cdot 7) - \sqrt{9}$	$120 = 1 \cdot (9 - 7 + 3)!$
$23 = 17 + 9 - 3$	$57 = 3 + 9(7 - 1)$	$89 = 9 \cdot (7 + 3) - 1$	$121 = 1 + (9 - 7 + 3)!$
$24 = (1^9 + 7) \cdot 3$	$58 = 9 + 7^{(3-1)}$	$90 = (1 \cdot 9) \cdot (7 + 3)$	122 = ✓
$25 = 9(3 - 1) + 7$	$59 = (9 \cdot 7) - 1 - 3$	$91 = 1 + 9 \cdot (7 + 3)$	123 =
$26 = 13(9 - 7)$	$60 = (9 \cdot 7) - (3 \cdot 1)$	$92 = 1 + 97 - 3!$	$124 = (13 \cdot 9) + 7$
$27 = 1^7 \cdot 9 \cdot 3$	$61 = 1 + (9 \cdot 7) - 3$	$93 = 97 - 1 - 3$	$125 = (\sqrt{9}! \cdot 7 \cdot 3) - 1$
$28 = 1^7 + (9 \cdot 3)$	$62 = (7 \cdot 9) - 1^3$	$94 = 97 - (1 \cdot 3)$	$126 = 1 \cdot \sqrt{9}! \cdot 7 \cdot 3$
$29 = 1 - 9 + 37$	$63 = 1^3 \cdot 9 \cdot 7$	$95 = 1 + 97 - 3$	$127 = (\sqrt{9}! \cdot 7 \cdot 3) + 1$
$30 = 3 \cdot (9 + 1^7)$	$64 = (19 \cdot 3) + 7$	$96 = (1 + \sqrt{9})! \cdot (7 - 3)$	$128 = 137 - 9$
$31 = 39 - 7 - 1$	$65 = (9 \cdot 7) - 1 + 3$	$97 = 1 - (9 + 7) \cdot 3!$	$129 = (1 + 7 - 3)! + 9$
$32 = 71 - 39$	$66 = 97 - 31$	$98 = (1 + 9 + 3) \cdot 7$	130 =
$33 = 31 + 9 - 7$			
$34 = 71 - (3 \cdot 9)$			

Better Late Than Never

John W. Langhaar sent in the following comment pertaining to **MAY 2** and the discussion of it in this column for October/November:

I interpret the discussion to say that the expression converges for $0 < x < \exp(i/e)$. That is not correct, as pointed out in *The Bent* of Tau Beta Pi, April, 1962. For $0 < x < \exp(-e)$, the expression oscillates without approaching a limit. The problem has more complications than meets the eye, and the behavior may be of interest to some readers.

Royce Zia and Lawrence Smith point out that the given expression converges for $(1/e)^e < x < e^{1/e}$, diverges for

Puzzle continued on p. 83

Although *Energy Through Nuclear Reactors* has more to say about nuclear physics fundamentals than reactor physics, enough of the latter is included to permit extrapolation of relevant information into the realm of current issues. For example, the fact that present day light-water-moderated (L.W.R.) reactors also convert uranium-238 into plutonium is clearly established, thereby making it unlikely that the reader will uncritically accept the misconstruction implied by some that the potential hazards of plutonium can somehow be avoided by not pursuing the breeder reactor. (As a point of fact, a 1,000-Mw.e. L.W.R. unit will produce approximately as much *new* plutonium each year as a comparably sized breeder reactor—about 200 kilograms.)

Once background information like this becomes more widely disseminated it may be possible to at least debate the real issues. In this spirit, the effort by Kuljian and Kramer must be welcomed despite occasional shortcomings, including a potentially misleading preoccupation with history at the expense of contemporaneity. In this category, precious space is devoted to reactor concepts no longer of interest; puzzling, in contrast, is the devotion of an entire chapter to high-energy particle physics in the expressed hope that it may someday lead to new energy sources in as yet unimaginable ways.

Finally, a taste of engineering flavor in a physics text will hopefully whet the appetite of both student and instructor for more. If so, they may integrate more economics with their physics and learn that everything that is scientifically feasible is not necessarily competitive in the energy marketplace. For example, the use of solar energy for central station power generation, often suggested as an alternative to nuclear power, would require development of solar collectors costing under $15/yd.^{2} (comparable to the cost of run-of-the-mill indoor carpeting) which could survive 30 years outdoors unaffected, with negligible maintenance. Or, conversely, students and their teachers could assuage whatever concern they might have over long-term disposal of radioactive wastes with the realization that a factor-of-ten increase in disposal costs could be absorbed with an almost negligible effect on the overall cost of nuclear power.

In the area of cost-risk-benefit analysis, readers will be edified to find that the hidden public health costs of nuclear power plants are a factor of ten less than those of fossil-fueled plants. They might even be able to temper an understandable initial disappointment at finding so little detail on fusion in *Energy Through Nuclear Reactors* by an appreciation of the formidable engineering obstacles to making fusion a viable energy source.

Michael J. Driscoll is a member of the Nuclear Engineering Department at M.I.T., where he teaches courses in nuclear power reactor design and engineering.

Boulding

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most a quarter of a billion people, and it still has capabilities for considerable population expansion. The invention of agriculture, perhaps stimulated by the niche-tightening forced on the human race by the last ice age, greatly expanded the human population potential. Metallurgy and now science have expanded it still further.

A Funny Kind of Quantity

How far these expansions can continue, we really do not know. It is our inevitable ignorance in this regard that makes growth policy so difficult. I have been teasing my fellow Coloradans lately by pointing out that the state, with its 2.25 million people, is almost exactly the same area as Italy with its 56 million, and that if Colorado were "polluted" with the likes of Florence, Pisa, Siena, Ravenna, and Rome (even if it couldn't have a Venice) it might not be a wholly disagreeable place in which to live.

This points towards the real problem of growth: that what is growing is not a number, or even a pile, but a complex organization and structure of many interrelated parts and large numbers of attributes. Any single scalar number, or even ordering, by which we measure growth, whether it is weight, volume, population, number of employees, net worth, number of members, or gross national product, is an abstraction from the vast complex society. It is the structure of the parts and attributes, not merely the overall size, that is relevant to the evaluation of them. That is, when we say "the bigger the better" or "the bigger the worse," what we are doing is comparing number or a size—which is derived by adding up some kind of common quality of all the various parts of the system—with an evaluation in terms of some kind of ultimate value. This value also depends on evaluating the various parts of the system, and perhaps their interrelationships, and getting some kind of number of size of total value as a result. This is what we usually mean by the quality of a system or structure.

The moral would seem to be: Take care of the quality and the quantity can take care of itself. We have to slip in a little rider that quality, after all, is nothing more than a funny kind of quantity with all the evaluative variables taken into consideration.

Kenneth E. Boulding is Professor of Economics at the University of Colorado and Director of the program on General, Social, and Economic Dynamics at the university's Institute of Behavioral Science. Born in England and educated at Oxford (B.A. 1931 and M.A. 1939), he is a former President of the American Economics Association.

Puzzle

Continued from p. 80

$x > e^{1/e}$, and oscillates for $0 < x < (1/e)^e$; and there is a unique negative solution of $x = -1$.

May 4 Frank Rubin has found that $x^5 + 6x^4 - 17x^3 - 51x^2 + 14x + 96$ has a root 3.141592653308, which differs from π by less than 3 in the 11th significant digit.

Solutions have been received for:

J/A1 Bob Baird

J/A2 Bob Baird, Sheldon Hoffman, and Ben Rouben

JN 3 Raymond Gaillard.

Speed Problem Solutions

FEB SD1 $3(11/36)^2\ (25/36) + (11/36)^3$, or approximately 0.22.

FEB SD2 Imagine a circle with a circumference of one mile centered at the South Pole. Start the hunter one mile north of the circle. He travels one mile south to the circle, one mile east around the circle, and one mile north to the point from which he started. There are an infinite number of starting points one mile north of the circle. Now imagine a circle with a circumference of half a mile centered at the South Pole. The hunter travels on the eastward point of his course walking around the circle twice. Repeat for 1/n mile circumference for n = 3, n = 4, n = 5, etc.

Allan J. Gottlieb studied mathematics at M.I.T. (S.B. 1967) and Brandeis (A.M. 1968, Ph.D. 1973) he is now Assistant Professor of Mathematics at York College of C.U.N.Y. Send problems, solutions, and comments to him at the Department of Mathematics, York College, 150-14 Jamaica Ave., Jamaica, N. Y., 11432.

Bridge

Continued from p. 81

West	North	East
	♠ Q J	
	♥ —	
	♦ x	
	♣ —	
♠ A (A)		♠ — (—)
♥ — (x)		♥ xxx (xx)
♦ 10 (—)		♦ — (10)
♣ x (x)		♣ — (—)
	(Gurwitz)	
	♠ —	
	♥ xx	
	♦ x	
	♣ —	

When Mike led a heart, West was destroyed. If West pitched the spade ace, Mike would ruff and lead the spade queen to force West's high trump. If West ruffed, Mike would discard a spade and cross ruff the last two cards. When West finally pitched a club. Mike ruffed the heart and scored the contract by ruffing a spade.

If East had possessed the long diamond (as shown above by the cards in parentheses), Mike would have ruffed the heart in dummy and won one of the last two tricks with his small trump.